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A new method to introduce additional separated variables for high-order binary constrained flows

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Abstract. Degrees of freedom for high-order binary constrained flows of soliton equations admitting 2×2 Lax matrices are $2N + k_0$. It is known that $N + k_0$ pairs of canonical separated variables for their separation of variables can be introduced directly via their Lax matrices. Here we propose a new method to introduce the additional N pairs of canonical separated variables and N additional separated equations. The Jacobi inversion problems for high-order binary constrained flows and for soliton equations are also established. This new method can be applied to all high-order binary constrained flows admitting 2×2 Lax matrices.

1. Introduction

For a finite-dimensional integrable Hamiltonian system (FDIHS), let m denote the number of degrees of freedom, and $P_i, i = 1, \dots, m$, be functionally independent integrals of motion in involution, the separation of variables means to construct m pairs of canonical separated variables $v_k, u_k, k = 1, \dots, m$ [1–3]

$$\{u_k, u_l\} = \{v_k, v_l\} = 0 \qquad \{v_k, u_l\} = \delta_{kl} \qquad k, l = 1, \dots, m \tag{1.1}$$

and m functions f_k such that

$$f_k(u_k, v_k, P_1, \dots, P_m) = 0 \qquad k = 1, \dots, m \tag{1.2}$$

which are called separated equations. Equations (1.2) give rise to an explicit factorization of the Liouville tori. For the FDIHSs with the Lax matrices admitting the r-matrices of the XXX, XXZ and XYZ type, there is a general approach to their separation of variables [1–6]. The corresponding separated equations enable us to express the generating function of canonical transformations in completely separated form as an Abelian integral on the associated invariant spectral curve. The resulting linearizing map is essentially the Abel map to the Jacobi variety of the spectral curve, thus providing a link with the algebro-geometric linearization methods given by [7–9].

The separation of variables for a FDIHS requires that the number of canonical separated variables u_k should be equal to the number of degrees of freedom m. In some cases, the number of u_k resulting from the normal method may be less than m and one needs to introduce some additional canonical separated variables. So far very few models in these cases have been studied. These cases remain to be a challenging problem [3].

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The separation of variables for constrained flows of soliton equations has been studied (see, for example, [4, 10–14]). In recent years binary constrained flows of soliton hierarchies have attracted attention (see, for example, [15–22]). However, the separation of variables for binary constrained flows has not been studied. The degree of freedom for high-order binary constrained flows admitting 2×2 Lax matrices $M = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}$ is a natural number $2N + k_0$. Via the Lax matrix M, $N + k_0$ pairs of canonical separated variables $u_1, \ldots, u_{N+k_0}$ can be introduced by the set of zeros of $B(\lambda)$ and $v_k = A(u_k)$, and $N + k_0$ separated equations can be found from the generation function of integrals of motions. In previous papers [23, 24] we presented a method with two different ways of determining N additional pairs of canonical separated variables and N additional separated equations for first binary constrained flows with $2N$ degrees of freedom. The main idea in [23, 24] is to construct two functions $\tilde{B}(\lambda)$ and $\tilde{A}(\lambda)$ and define $u_{N+1}, \ldots, u_{2N}$ by the set of zeros of $\tilde{B}(\lambda)$ and $v_{N+k} = \tilde{A}(u_{N+k})$. The ways of constructing $\tilde{B}(\lambda)$ and $\tilde{A}(\lambda)$ in [23, 24] are somewhat different. Here we propose a completely different method from that in [23, 24] to introduce the additional N separated variables and N separated equations for high-order binary constrained flows with $2N + k_0$ degrees of freedom. It is observed that the introduction of v_k has some link with integrals of motion and should lead to the separated equations. We find that there are N integrals of motion $P_{N+k_0+1}, \ldots, P_{2N+k_0}$ among the $2N+k_0$ integrals of motion for the high-order binary constrained flows which commute with $A(\lambda)$ and $B(\lambda)$. This observation and property stimulate us to use the additional integrals of motion directly to define both the N pairs of additional separated variables and N separated equations by $v_{N+k_0+j} = P_{N+k_0+j}$, $j = 1, \ldots, N$. Then we can find the conjugated variables $u_{N+k_0+j}, 1, \ldots, N$, commuting with $A(\lambda)$ and $B(\lambda)$. In contrast to the method in [23, 24], this method is easier to apply to the high-order binary constrained flows.

We will also present the separation of variables of soliton equations. The first step is to factorize (1 + 1)-dimensional soliton equations into two commuting x- and t-FDIHSs via high-order binary constrained flows, namely the x and t dependences of the soliton equations are separated by the x- and t-FDIHSs obtained from the x and t binary constrained flows. The second step is to produce separation of variables for the x- and t-FDIHDs. Finally, combining the factorization of soliton equations with the Jacobi inversion problems for x- and t-FDIHSs enables us to establish the Jacobi inversion problems for soliton equations. If the Jacobi inversion problem can be solved by the Jacobi inversion technique [7], one can obtain the solution in terms of the Riemann-theta function for soliton equations. We illustrate the method by KdV, AKNS and Kaup–Newell (KN) hierarchies. The paper is organized as follows.

In section 2, we first recall the high-order binary constrained flows and factorization of the KdV hierarchy. Then we propose a method for introducing the N pairs of additional separated variables. We illustrate the method by both first binary constrained flow and second binary constrained flow. Finally, we present the separation of variables for the KdV hierarchy. In sections 3 and 4, the method is applied to the AKNS hierarchy and KN hierarchy, respectively. In fact, this method can be applied to all high-order binary constrained flows and other soliton hierarchies admitting 2×2 Lax pairs.

2. Separation of variables for the KdV equations

We first recall the high-order binary constrained flows of the KdV hierarchy.

2.1. High-order binary constrained flows of the KdV hierarchy

Consider the Schrödinger equation [25]

$$\phi_{xx} + (\lambda + u)\,\phi = 0$$

which is equivalent to the following spectral problem:

$$\phi_x = U(u,\lambda)\,\phi \qquad U(u,\lambda) = \begin{pmatrix} 0 & 1 \\ -\lambda - u & 0 \end{pmatrix} \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{2.1}$$

Take the time evolution law of ϕ as

$$\phi_{t_n} = V^{(n)}(u,\lambda)\,\phi \tag{2.2}$$

where

$$\begin{aligned} &V^{(n)}(u,\lambda) = \sum_{i=0}^{n+1} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{n+1-i} + \begin{pmatrix} 0 & 0 \\ b_{n+2} & 0 \end{pmatrix} \\ &a_0 = b_0 = 0 \qquad c_0 = -1 \qquad a_1 = 0 \qquad b_1 = 1 \\ &b_{k+1} = Lb_k = -\tfrac{1}{2}L^{k-1}u \qquad a_k = -\tfrac{1}{2}b_{k,x} \\ &c_k = -\tfrac{1}{2}b_{k,xx} - b_{k+1} - b_k u \qquad k = 1,2,\ldots \\ &L = -\tfrac{1}{4}\partial^2 - u + \tfrac{1}{2}\partial^{-1}u_x \qquad \partial = \partial_x \qquad \partial^{-1}\partial = \partial\partial^{-1} = 1. \end{aligned} \tag{2.3}$$

The compatibility condition of (2.1) and (2.2) gives rise to the nth KdV equation which can be written as an infinite-dimensional Hamiltonian system [25]

$$u_{t_n} = -2b_{n+2,x} = \partial L^n u = \partial \frac{\delta H_n}{\delta u} \tag{2.4}$$

with the Hamiltonian $H_n = 4b_{n+3}/(2n+3)$ and $\delta H_n/\delta u = -2b_{n+2}$.

For $n = 1$ we have

$$\phi_{t_1} = V^{(1)}(u,\lambda)\,\phi \qquad V^{(1)} = \begin{pmatrix} \frac{1}{4}u_x & \lambda - \frac{1}{2}u \\ -\lambda^2 - \frac{1}{2}u\lambda + \frac{1}{4}u_{xx} + \frac{1}{2}u^2 & -\frac{1}{4}u_x \end{pmatrix} \tag{2.5}$$

and equation (2.4) for $n = 1$ is the well known KdV equation

$$u_{t_1} = -\tfrac{1}{4}(u_{xxx} + 6uu_x). \tag{2.6}$$

The adjoint spectral problem reads

$$\psi_x = -U^T(u,\lambda)\psi \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.7}$$

By means of the formula in [26], we have

$$\frac{\delta\lambda}{\delta u} = \mathrm{Tr}\left[\begin{pmatrix} \phi_1\psi_1 & \phi_1\psi_2 \\ \phi_2\psi_1 & \phi_2\psi_2 \end{pmatrix} \frac{\partial U(u,\lambda)}{\partial u}\right] = -\psi_2\phi_1.$$

According to [15–22], the high-order binary x-constrained flows of the KdV hierarchy (2.4) consist of the equations obtained from the spectral problem (2.1) and the adjoint spectral problem (2.7) for N distinct real numbers λ_j and the restriction of the variational derivatives for the conserved quantities H_{k_0} (for any fixed k_0) and λ_j:

$$\Phi_{1,x} = \Phi_2 \qquad \Phi_{2,x} = -\Lambda\Phi_1 - u\Phi_1 \tag{2.8a}$$

$$\Psi_{1,x} = \Lambda\Psi_2 + u\Psi_2 \qquad \Psi_{2,x} = -\Psi_1 \tag{2.8b}$$

$$\frac{\delta H_{k_0}}{\delta u} - \sum_{j=1}^{N} \frac{\delta\lambda_j}{\delta u} = -2b_{k_0+2} + \langle\Psi_2, \Phi_1\rangle = 0. \tag{2.8c}$$

Hereafter we denote the inner product in $\mathbb{R}^N$ by $\langle\cdot\,,\cdot\rangle$ and

$$\Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^T \qquad \Psi_i = (\psi_{i1}, \ldots, \psi_{iN})^T \qquad i = 1, 2$$

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N).$$

The binary t_n-constrained flows of the KdV hierarchy (2.4) are defined by the replicas of (2.2) and its adjoint system for N distinct real number λ_j,

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix}_{t_n} = V^{(n)}(u, \lambda_j) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \end{pmatrix} \qquad \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_{t_n} = -(V^{(n)}(u, \lambda_j))^T \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}$$

$$j = 1, \ldots, N \tag{2.9a}$$

as well as the nth KdV equation itself (2.4) in the case of the higher-order constraint for $k_0 \geqslant 1$

$$u_{t_n} = -2b_{n+2,x}. \tag{2.9b}$$

2.1.1. For $k_0 = 0$ we have

$$b_2 = -\tfrac{1}{2}u = \tfrac{1}{2}\langle\Psi_2, \Phi_1\rangle \qquad \text{i.e.} \quad u = -\langle\Psi_2, \Phi_1\rangle. \tag{2.10}$$

By substituting (2.10), (2.8a) and (2.8b) becomes a finite-dimensional Hamiltonian system (FDHS) [18]

$$\Phi_{1x} = \frac{\partial F_1}{\partial \Psi_1} \qquad \Phi_{2x} = \frac{\partial F_1}{\partial \Psi_2} \qquad \Psi_{1x} = -\frac{\partial F_1}{\partial \Phi_1} \qquad \Psi_{2x} = -\frac{\partial F_1}{\partial \Phi_2}$$
$$F_1 = \langle\Psi_1, \Phi_2\rangle - \langle\Lambda\Psi_2, \Phi_1\rangle + \tfrac{1}{2}\langle\Psi_2, \Phi_1\rangle^2. \tag{2.11}$$

Under the constraint (2.10) and the x-FDHS (2.11), the binary t_1-constrained flow obtained from (2.9a) with $V^{(1)}$ given by (2.5) can also be written as a t_1-FDHS

$$\Phi_{1,t_1} = \frac{\partial F_2}{\partial \Psi_1} \qquad \Phi_{2,t_1} = \frac{\partial F_2}{\partial \Psi_2} \qquad \Psi_{1,t_1} = -\frac{\partial F_2}{\partial \Phi_1} \qquad \Psi_{2,t_1} = -\frac{\partial F_2}{\partial \Phi_2}$$
$$F_2 = -\langle\Lambda^2\Psi_2, \Phi_1\rangle + \langle\Lambda\Psi_1, \Phi_2\rangle + \tfrac{1}{2}\langle\Psi_2, \Phi_1\rangle\langle\Lambda\Psi_2, \Phi_1\rangle$$
$$+\tfrac{1}{2}\langle\Psi_2, \Phi_1\rangle\langle\Psi_1, \Phi_2\rangle + \tfrac{1}{8}(\langle\Psi_2, \Phi_2\rangle - \langle\Psi_1, \Phi_1\rangle)^2. \tag{2.12}$$

The Lax representation for the x-constrained flow (2.8) and the t_n-constrained flow (2.9) can be deduced from the adjoint representation of (2.1) and (2.2) by using the method in [27, 28]

$$M_x = [\tilde{U}, M] \qquad M_{t_n} = [\tilde{V}^{(n)}, M] \tag{2.13}$$

where $\tilde{U}$ and $\tilde{V}^{(n)}$ are obtained from U and $V^{(n)}$ under the system (2.8), and the Lax matrix M is of the form

$$M = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}.$$

The Lax matrix M for x-FDHS (2.11) and t_1-FDHS (2.12) is given by

$$A(\lambda) = \frac{1}{4}\sum_{j=1}^{N}\frac{\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j}}{\lambda - \lambda_j} \qquad B(\lambda) = 1 + \frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{2j}\phi_{1j}}{\lambda - \lambda_j}$$
$$C(\lambda) = -\lambda + \tfrac{1}{2}\langle\Psi_2, \Phi_1\rangle + \tfrac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}\phi_{2j}}{\lambda - \lambda_j}. \tag{2.14}$$

The generating function of integrals of motion for (2.11) and (2.12) yields

$$A^2(\lambda) + B(\lambda)\,C(\lambda) \equiv P(\lambda) = -\lambda + \sum_{j=1}^{N}\left[\frac{P_j}{\lambda - \lambda_j} + \frac{P_{N+j}^2}{(\lambda - \lambda_j)^2}\right] \tag{2.15}$$

where $P_1, \ldots, P_{2N}$ are independent integrals of motion for the FDHSs (2.11) and (2.12)

$$P_j = \tfrac{1}{2}\psi_{1j}\phi_{2j} + \left(-\tfrac{1}{2}\lambda_j + \tfrac{1}{4}\langle\Psi_2, \Phi_1\rangle\right)\psi_{2j}\phi_{1j}$$
$$+\frac{1}{8}\sum_{k\neq j}\frac{1}{\lambda_j - \lambda_k}[(\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j})(\psi_{1k}\phi_{1k} - \psi_{2k}\phi_{2k}) + 4\psi_{1j}\phi_{2j}\psi_{2k}\phi_{1k}] \tag{2.16a}$$

$$P_{N+j} = \tfrac{1}{4}(\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}) \qquad j = 1, \ldots, N. \tag{2.16b}$$

It is easy to verify that

$$F_1 = 2\sum_{j=1}^{N} P_j \qquad F_2 = 2\sum_{j=1}^{N}(\lambda_j P_j + P_{N+j}^2). \tag{2.17}$$

With respect to the standard Poisson bracket

$$\{f, g\} = \sum_{j=1}^{N}\left(\frac{\partial f}{\partial\psi_{1j}}\frac{\partial g}{\partial\phi_{1j}} + \frac{\partial f}{\partial\psi_{2j}}\frac{\partial g}{\partial\phi_{2j}} - \frac{\partial f}{\partial\phi_{1j}}\frac{\partial g}{\partial\psi_{1j}} - \frac{\partial f}{\partial\phi_{2j}}\frac{\partial g}{\partial\psi_{2j}}\right) \tag{2.18}$$

by calculating formulae like (2.31), it is easy to verify that

$$\{A^2(\lambda) + B(\lambda)\,C(\lambda), A^2(\mu) + B(\mu)\,C(\mu)\} = 0 \tag{2.19}$$

which implies that $P_1, \ldots, P_{2N}$ are in involution, equations (2.11) and (2.12) are FDIHSs and commute with each other. The construction of (2.11) and (2.12) ensures that if $(\Psi_1, \Psi_2, \Phi_1, \Phi_2)$ satisfies the FDIHSs (2.11) and (2.12) simultaneously, then u defined by (2.10) solves the KdV equation (2.6).

Set

$$A^2(\lambda) + B(\lambda)\,C(\lambda) = \lambda\sum_{k=0}^{\infty}\tilde{F}_k\lambda^{-k} \tag{2.20}$$

where $\tilde{F}_k, k = 1, 2, \ldots,$ are also integrals of motion for both the x-FDHSs (2.11) and the t_n-binary constrained flows (2.9). Comparing (2.20) with (2.15), one obtains

$$\tilde{F}_0 = -1 \qquad \tilde{F}_1 = 0 \qquad \tilde{F}_k = \sum_{j=1}^{N}\left[\lambda_j^{k-2}P_j + (k-2)\lambda_j^{k-3}P_{N+j}^2\right] \qquad k = 2, 3, \ldots. \tag{2.21}$$

By employing the method in [28, 29], the t_n-FDIHS obtained from the t_n-binary constrained flow (2.9) is found to be of the form

$$\Phi_{1,t_n} = \frac{\partial F_{n+1}}{\partial \Psi_1} \qquad \Phi_{2,t_n} = \frac{\partial F_{n+1}}{\partial \Psi_2} \qquad \Psi_{1,t_n} = -\frac{\partial F_{n+1}}{\partial \Phi_1} \qquad \Psi_{2,t_n} = -\frac{\partial F_{n+1}}{\partial \Phi_2} \tag{2.22a}$$

$$F_{n+1} = \sum_{m=0}^{n} (\tfrac{1}{2})^{m-1} \frac{\alpha_m}{m+1} \sum_{l_1+\cdots+l_{m+1}=n+2} \tilde{F}_{l_1} \ldots \tilde{F}_{l_{m+1}} \tag{2.22b}$$

where $l_1 \geqslant 1, \ldots, l_{m+1} \geqslant 1, \alpha_0 = 1, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{3}{2}$, and [28, 29]

$$\alpha_m = 2\alpha_{m-1} + \sum_{l=1}^{m-2} \alpha_l \alpha_{m-l-1} - \tfrac{1}{2} \sum_{l=1}^{m-1} \alpha_l \alpha_{m-l} \qquad m \geqslant 3. \tag{2.22c}$$

The nth KdV equation (2.4) is factorized by the x-FDIHS (2.11) and the t_n-FDIHS (2.22).

2.1.2. For $k_0 = 1$ one obtains

$$b_3 = \tfrac{1}{8}(u_{xx} + 3u^2) = \tfrac{1}{2}\langle \Psi_2, \Phi_1 \rangle. \tag{2.23}$$

By introducing $q = u, p = \frac{1}{4}u_x$, equations (2.8$a$), (2.8$b$) and (2.23) can be written as a x-FDHS

$$\begin{gathered} \Phi_{ix} = \frac{\partial F_1}{\partial \Psi_i} \qquad \Psi_{ix} = -\frac{\partial F_1}{\partial \Phi_i} \qquad i = 1, 2 \qquad q_x = \frac{\partial F_1}{\partial p} \qquad p_x = -\frac{\partial F_1}{\partial q} \\ F_1 = -\langle \Lambda \Psi_2, \Phi_1 \rangle + \langle \Psi_1, \Phi_2 \rangle - q\langle \Psi_2, \Phi_1 \rangle + 2p^2 + \tfrac{1}{4}q^3. \end{gathered} \tag{2.24}$$

Under the constraint (2.23), $V^{(1)}$ becomes

$$\tilde{V}^{(1)} = \begin{pmatrix} p & \lambda - \frac{1}{2}q \\ -\lambda^2 - \frac{1}{2}q\lambda + \langle \Psi_2, \Phi_1 \rangle - \frac{1}{4}q^2 & -p \end{pmatrix}. \tag{2.25}$$

Under the constraint (2.23) and the x-FDHS (2.24), the binary t_1-constrained flow consists of (2.9a) with $V^{(1)}$ replaced by $\tilde{V}^{(1)}$ and (2.9b) given by (2.6) can also be written as a t_1-FDHS

$$\begin{gathered} \Phi_{it_1} = \frac{\partial F_2}{\partial \Psi_i} \qquad \Psi_{it_1} = -\frac{\partial F_2}{\partial \Phi_i} \qquad i = 1, 2 \qquad q_{t_1} = \frac{\partial F_2}{\partial p} \qquad p_{t_1} = -\frac{\partial F_2}{\partial q} \\ F_2 = -\langle \Lambda^2 \Psi_2, \Phi_1 \rangle + \langle \Lambda \Psi_1, \Phi_2 \rangle - \tfrac{1}{2}q\langle \Lambda \Psi_2, \Phi_1 \rangle - \tfrac{1}{2}q\langle \Psi_1, \Phi_2 \rangle \\ + p\langle \Psi_1, \Phi_1 \rangle - p\langle \Psi_2, \Phi_2 \rangle + \tfrac{1}{2}\langle \Psi_2, \Phi_1 \rangle^2 - \tfrac{1}{4}q^2\langle \Psi_2, \Phi_1 \rangle. \end{gathered} \tag{2.26}$$

The Lax representations for the x-FDHS (2.24) and the t_1-FDHS (2.26), which can be deduced from the adjoint representation of (2.1) and (2.2), are given by (2.13) with $\tilde{V}^{(1)}$ defined by (2.25) and $\tilde{U}$ obtained from U by using q instead of u as well as M given by

$$\begin{gathered} A(\lambda) = p + \frac{1}{4} \sum_{j=1}^{N} \frac{\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j}}{\lambda - \lambda_j} \qquad B(\lambda) = \lambda - \tfrac{1}{2}q + \tfrac{1}{2} \sum_{j=1}^{N} \frac{\psi_{2j}\phi_{1j}}{\lambda - \lambda_j} \\ C(\lambda) = -\lambda^2 - \tfrac{1}{2}q\lambda + \tfrac{1}{2}\langle \Psi_2, \Phi_1 \rangle - \tfrac{1}{4}q^2 + \tfrac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}\phi_{2j}}{\lambda - \lambda_j}. \end{gathered} \tag{2.27}$$

The generating function of integrals of motion for (2.24) and (2.26) yields

$$A^2(\lambda) + B(\lambda)\,C(\lambda) \equiv P(\lambda) = -\lambda^3 + P_0 + \sum_{j=1}^{N} \left[\frac{P_j}{\lambda - \lambda_j} + \frac{P_{N+j}^2}{(\lambda - \lambda_j)^2} \right] \tag{2.28}$$

where $P_0, \ldots, P_{2N}$ are independent integrals of motion for the FDHSs (2.24) and (2.26) and $P_0 = \frac{1}{2}F_1$,

$$P_j = -\tfrac{1}{2}\lambda_j^2\psi_{2j}\phi_{1j} + \tfrac{1}{2}\lambda_j\psi_{1j}\phi_{2j} - \tfrac{1}{4}\lambda_j q\psi_{2j}\phi_{1j} - \tfrac{1}{4}q\psi_{1j}\phi_{2j} + \tfrac{1}{2}p(\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j})$$
$$+\tfrac{1}{4}\big(\langle\Psi_2, \Phi_1\rangle - \tfrac{1}{2}q^2\big)\psi_{2j}\phi_{1j} + \tfrac{1}{8}\sum_{k\neq j}\frac{1}{\lambda_j - \lambda_k}$$
$$\times[(\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j})(\psi_{1k}\phi_{1k} - \psi_{2k}\phi_{2k}) + 4\psi_{1j}\phi_{2j}\psi_{2k}\phi_{1k}] \qquad (2.29a)$$

$$P_{N+j} = \tfrac{1}{4}(\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}) \qquad j = 1, \ldots, N. \qquad (2.29b)$$

We have

$$F_1 = 2P_0 \qquad F_2 = 2\sum_{j=1}^{N} P_j. \qquad (2.30)$$

Similarly, it can be shown that (2.24) and (2.26) are FDIHSs and commute with each other. The KdV equation (2.6) is factorized by x-FDIHS (2.24) and t_1-FDIHS (2.26). If $(\Psi_1, \Psi_2, p, \Phi_1, \Phi_2, q)$ satisfies the FDIHSs (2.24) and (2.26) simultaneously, then $u = q$ solves the KdV equation (2.6).

2.2. The separation of variables for the KdV equations

2.2.1. For the case $k_0 = 0$ we first consider the separation of variables for FDIHSs (2.11) and (2.12). With respect to the standard Poisson bracket (2.18), it is found that for $A(\lambda)$ and $B(\lambda)$ given by (2.14) we have

$$\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = 0 \qquad \{A(\lambda), B(\mu)\} = \frac{1}{2(\lambda - \mu)}[B(\mu) - B(\lambda)]. \qquad (2.31)$$

An effective way to introduce the separated variables v_k, u_k and to obtain the separated equations is to use the Lax matrix M and the generating function of integrals of motion. The commutator relations (2.31) and equation (2.15) enable us to define the first N pairs of the canonical variables $u_1, \ldots, u_N$ by the set of zeros of $B(\lambda)$ [1–3]

$$B(\lambda) = 1 + \frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{2j}\phi_{1j}}{\lambda - \lambda_j} = \frac{R(\lambda)}{K(\lambda)} \qquad (2.32a)$$

where

$$R(\lambda) = \prod_{k=1}^{N}(\lambda - u_k) \qquad K(\lambda) = \prod_{k=1}^{N}(\lambda - \lambda_k)$$

and $v_1, \ldots, v_N$ by

$$v_k = 2A(u_k) \qquad k = 1, \ldots, N. \qquad (2.32b)$$

The commutator relations (2.31) guarantee that $u_1, \ldots, u_N$ and $v_1, \ldots, v_N$ satisfy the canonical conditions (1.1) [1–3]. Then substituting u_k into (2.15) gives rise to the first N separated equations

$$v_k = 2A(u_k) = 2\sqrt{P(u_k)} = 2\sqrt{-u_k + \sum_{j=1}^{N}\left[\frac{P_j}{u_k - \lambda_j} + \frac{P_{N+j}^2}{(u_k - \lambda_j)^2}\right]} \qquad k = 1, \ldots, N. \qquad (2.33)$$

The FDIHSs (2.11) and (2.12) have $2N$ degrees of freedom, we need to introduce the other N pairs of canonical variables $v_k, u_k, k = N+1, \ldots, 2N$. Notice that P_{N+j} given by (2.16*b*) are integrals of motion for the FDIHSs (2.11) and (2.12), and satisfy

$$\{B(\lambda), P_{N+j}\} = \{A(\lambda), P_{N+j}\} = 0. \tag{2.34}$$

Thus we may define

$$v_{N+j} = 2P_{N+j} = \tfrac{1}{2}(\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}) \qquad j = 1, \ldots, N \tag{2.35a}$$

which also give rise to the separated equations. It is easy to see that if we take

$$u_{N+j} = \ln \frac{\phi_{1j}}{\psi_{2j}} \qquad j = 1, \ldots, N \tag{2.35b}$$

then

$$\{v_{N+j}, u_{N+k}\} = \delta_{jk} \qquad \{v_{N+j}, v_{N+k}\} = \{u_{N+j}, u_{N+k}\} = 0 \qquad j, k = 1, \ldots, N \tag{2.36}$$

$$\{B(\lambda), u_{N+j}\} = \{A(\lambda), u_{N+j}\} = \{B(\lambda), v_{N+j}\} = \{A(\lambda), v_{N+j}\} = 0. \tag{2.37}$$

We have the following proposition.

Proposition 1. *Assume that $\lambda_j, \phi_{ij}, \psi_{ij} \in \mathbb{R}, i = 1, 2, j = 1, \ldots, N$. Introduce the separated variables $u_1, \ldots, u_{2N}$ and $v_1, \ldots, v_{2N}$ by (2.32) and (2.35). If $u_1, \ldots, u_N$, are single zeros of $B(\lambda)$, then $v_1, \ldots, v_{2N}$ and $u_1, \ldots, u_{2N}$ are canonically conjugated, i.e. they satisfy (1.1).*

Proof. By following the similar method in [1–6, 23, 24], it is easy to show that $v_1, \ldots, v_N$ and $u_1, \ldots, u_N$ satisfy (1.1). Notice that $B'(u_k) \neq 0$. Hereafter the prime denotes the differentiation with respect to λ. It follows from (2.36) and (2.37) that

$$\begin{aligned} 0 &= \{u_{N+k}, B(u_l)\} = B'(u_l)\{u_{N+k}, u_l\} + \{u_{N+k}, B(\mu)\}|_{\mu=u_l} = B'(u_l)\{u_{N+k}, u_l\} \\ \{v_k, u_{N+l}\} &= 2\{A(u_k), u_{N+l}\} \\ &= 2A'(u_k)\{u_k, u_{N+l}\} + \{A(\lambda), u_{N+l}\}|_{\lambda=u_k} = 2A'(u_k)\{u_k, u_{N+l}\} \end{aligned} \tag{2.38}$$

which leads to $\{u_{N+k}, u_l\} = \{u_{N+k}, v_l\} = 0$. Similarly, we can show that $\{v_{N+k}, u_l\} = \{v_{N+k}, v_l\} = 0$. These together with (2.36) complete the proof. □

It follows from (2.32*a*) and (2.35*b*) that

$$\begin{aligned} &u = -\langle\Psi_2, \Phi_1\rangle = 2\sum_{j=1}^{N}(u_j - \lambda_j) \\ &\psi_{2j}\phi_{1j} = 2\frac{R(\lambda_j)}{K'(\lambda_j)} \qquad \frac{\phi_{1j}}{\psi_{2j}} = \mathrm{e}^{u_{N+j}} \qquad j = 1, \ldots, N \end{aligned} \tag{2.39}$$

or

$$\phi_{1j} = \sqrt{\frac{2R(\lambda_j)\,\mathrm{e}^{u_{N+j}}}{K'(\lambda_j)}} \qquad \psi_{2j} = \sqrt{\frac{2R(\lambda_j)\,\mathrm{e}^{-u_{N+j}}}{K'(\lambda_j)}} \qquad j = 1, \ldots, N. \tag{2.40}$$

The separated equations are given by (2.33) and (2.35*a*). Replacing v_k by the partial derivative $\partial S/\partial u_k$ of the generating function S of the canonical transformation and interpreting P_i as integration constants, equations (2.33) and (2.35*a*) give rise to the Hamilton–Jacobi equations which are completely separable and may be integrated to give the completely separated solution

$$S(u_1, \ldots, u_{2N}) = \sum_{k=1}^{N} \left[\int^{u_k} 2\sqrt{P(\lambda)}\, \mathrm{d}\lambda + 2P_{N+k}u_{N+k} \right]. \tag{2.41}$$

The linearizing coordinates are then

$$Q_i = \frac{\partial S}{\partial P_i} = \sum_{k=1}^{N} \int^{u_k} \frac{1}{(\lambda - \lambda_i)\sqrt{P(\lambda)}}\, \mathrm{d}\lambda \qquad i = 1, \ldots, N \tag{2.42a}$$

$$Q_{N+i} = \frac{\partial S}{\partial P_{N+i}} = 2\sum_{k=1}^{N} \int^{u_k} \frac{P_{N+i}}{(\lambda - \lambda_i)^2\sqrt{P(\lambda)}}\, \mathrm{d}\lambda + 2u_{N+i} \qquad i = 1, \ldots, N. \tag{2.42b}$$

By using (2.17), the linear flow induced by (2.11) is then given by

$$Q_i = \gamma_i + x\frac{\partial F_1}{\partial P_i} = \gamma_i + 2x \qquad Q_{N+i} = 2\gamma_{N+i} + x\frac{\partial F_1}{\partial P_{N+i}} = 2\gamma_{N+i} \qquad i = 1, \ldots, N. \tag{2.43}$$

Hereafter $\gamma_i, i = 1, \ldots, 2N$, are arbitrary constants. Combining equation (2.42) with equation (2.43) leads to the Jacobi inversion problem for the FDIHS (2.11)

$$\sum_{k=1}^{N} \int^{u_k} \frac{1}{(\lambda - \lambda_i)\sqrt{P(\lambda)}}\, \mathrm{d}\lambda = \gamma_i + 2x \qquad i = 1, \ldots, N \tag{2.44a}$$

$$\sum_{k=1}^{N} \int^{u_k} \frac{P_{N+i}}{(\lambda - \lambda_i)^2\sqrt{P(\lambda)}}\, \mathrm{d}\lambda + u_{N+i} = \gamma_{N+i} \qquad i = 1, \ldots, N. \tag{2.44b}$$

If, by using the Jacobi inversion technique [7], $\phi_{1j}, \psi_{2j}, \langle \Psi_2, \Phi_1 \rangle$ given by (2.39) and (2.40) can be obtained from (2.44), then ϕ_{2j}, ψ_{1j} can be found from the first and the last equation in (2.11) by an algebraic calculation, respectively. The $(\phi_{1j}, \phi_{2j}, \psi_{1j}, \psi_{2j})$ provides the solution to the FDIHS (2.11).

By using (2.17), the linear flow induced by (2.12) is then given by

$$\begin{aligned} Q_i &= \bar{\gamma}_i + \frac{\partial F_2}{\partial P_i}t_1 = \bar{\gamma}_i + 2\lambda_i t_1 \\ Q_{N+i} &= 2\bar{\gamma}_{N+i} + \frac{\partial F_2}{\partial P_{N+i}}t_1 = 2\bar{\gamma}_{N+i} + 4P_{N+i}t_1 \qquad i = 1, \ldots, N \end{aligned} \tag{2.45}$$

where $\bar{\gamma}_i$ are arbitrary constants. Combining equation (2.42) with equation (2.45) leads to the Jacobi inversion problem for the FDIHS (2.12)

$$\sum_{k=1}^{N} \int^{u_k} \frac{1}{(\lambda - \lambda_i)\sqrt{P(\lambda)}}\, \mathrm{d}\lambda = \bar{\gamma}_i + 2\lambda_i t_1 \qquad i = 1, \ldots, N \tag{2.46a}$$

$$\sum_{k=1}^{N} \int^{u_k} \frac{P_{N+i}}{(\lambda - \lambda_i)^2\sqrt{P(\lambda)}}\, \mathrm{d}\lambda + u_{N+i} = \bar{\gamma}_{N+i} + 2P_{N+i}t_1 \qquad i = 1, \ldots, N. \tag{2.46b}$$

Finally, since the KdV equation (2.6) is factorized by the FDIHSs (2.11) and (2.12), combining equation (2.44) with equation (2.46) gives rise to the Jacobi inversion problem for the KdV equation (2.6)

$$\sum_{k=1}^{N}\int^{u_k}\frac{1}{(\lambda-\lambda_i)\sqrt{P(\lambda)}}\,\mathrm{d}\lambda=\gamma_i+2x+2\lambda_i t_1 \qquad i=1,\ldots,N \tag{2.47a}$$

$$\sum_{k=1}^{N}\int^{u_k}\frac{P_{N+i}}{(\lambda-\lambda_i)^2\sqrt{P(\lambda)}}\,\mathrm{d}\lambda+u_{N+i}=\gamma_{N+i}+2P_{N+i}t_1 \qquad i=1,\ldots,N. \tag{2.47b}$$

If, by using the Jacobi inversion technique [7], u is given by (2.39) can be found in terms of Riemann theta functions by solving (2.47), then u provides the solution of the KdV equation (2.6).

In general, since the nth KdV equation (2.4) is factorized by the x-FDIHS (2.11) and the t_n-FDIHS (2.22), the above procedure can be applied to find the Jacobi inversion problem for the nth KdV equation (2.4). We have the following proposition.

Proposition 2. *The Jacobi inversion problem for the nth KdV equation (2.4) is given by*

$$\sum_{k=1}^{N}\int^{u_k}\frac{1}{(\lambda-\lambda_i)\sqrt{P(\lambda)}}\,\mathrm{d}\lambda=\gamma_i+2x$$
$$+t_n\sum_{m=0}^{n}(\tfrac{1}{2})^{m-1}\alpha_m\sum_{l_1+\cdots+l_{m+1}=n+2}\lambda_i^{l_{m+1}-2}\tilde{F}_{l_1}\ldots\tilde{F}_{l_m} \qquad i=1,\ldots,N$$

$$\sum_{k=1}^{N}\int^{u_k}\frac{P_{N+i}}{(\lambda-\lambda_i)^2\sqrt{P(\lambda)}}\,\mathrm{d}\lambda+u_{N+i}=\gamma_{N+i}+t_n\sum_{m=0}^{n}(\tfrac{1}{2})^{m-2}\alpha_m$$
$$\times\sum_{l_1+\cdots+l_{m+1}=n+2}(l_{m+1}-2)\lambda_i^{l_{m+1}-3}P_{N+i}\tilde{F}_{l_1}\ldots\tilde{F}_{l_m} \qquad i=1,\ldots,N$$

where $l_1\geqslant 1,\ldots,l_{m+1}\geqslant 1$ and $\tilde{F}_{l_1},\ldots\tilde{F}_{l_m}$, are given by (2.21).

2.2.2. For the case $k_0=1$ we now consider the separation of variables for FDIHSs (2.24) and (2.26). With respect to the standard Poisson bracket, it is found that $A(\lambda)$ and $B(\lambda)$ given by (2.27) also satisfy the commutator relation (2.31). In the same way, the first $N+1$ pairs of the canonical variables $u_1,\ldots,u_{N+1}$ can be introduced by the set of zeros of $B(\lambda)$

$$B(\lambda)=\lambda-\tfrac{1}{2}q+\tfrac{1}{2}\sum_{j=1}^{N}\frac{\psi_{2j}\phi_{1j}}{\lambda-\lambda_j}=\frac{R(\lambda)}{K(\lambda)} \tag{2.48a}$$

where

$$R(\lambda)=\prod_{k=1}^{N+1}(\lambda-u_k) \qquad K(\lambda)=\prod_{k=1}^{N}(\lambda-\lambda_k)$$

and $v_1,\ldots,v_{N+1}$ by

$$v_k=2A(u_k) \qquad k=1,\ldots,N+1. \tag{2.48b}$$

Then substituting u_k into (2.28) gives rise to the first $N+1$ separated equations

$$v_k=2A(u_k)=2\sqrt{P(u_k)}=2\sqrt{-u_k^3+P_0+\sum_{j=1}^{N}\left[\frac{P_j}{u_k-\lambda_j}+\frac{P_{N+j}^2}{(u_k-\lambda_j)^2}\right]}$$
$$k=1,\ldots,N+1. \tag{2.49}$$

The additional N pairs of canonical variables can also be defined in the same way

$$v_{N+1+j} = 2P_{N+j} = \frac{1}{2}(\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}) \qquad j = 1, \ldots, N \tag{2.50a}$$

$$u_{N+1+j} = \ln \frac{\phi_{1j}}{\psi_{2j}} \qquad j = 1, \ldots, N. \tag{2.50b}$$

In the same way we can show the following proposition.

Proposition 3. *Assume that $\lambda_j, \phi_{ij}, \psi_{ij} \in \mathbb{R}, i = 1, 2, j = 1, \ldots, N$. Introduce the separated variables $u_1, \ldots, u_{2N+1}$ and $v_1, \ldots, v_{2N+1}$ by (2.48) and (2.50). If $u_1, \ldots, u_{N+1}$, are single zeros of $B(\lambda)$, then $v_1, \ldots, v_{2N+1}$ and $u_1, \ldots, u_{2N+1}$ are canonically conjugated, i.e. they satisfy (1.1).*

It follows from (2.48) and (2.50) that

$$u = q = 2\sum_{j=1}^{N+1} u_j - 2\sum_{j=1}^{N} \lambda_j \tag{2.51a}$$

$$\phi_{1j} = \sqrt{\frac{2R(\lambda_j)\,\mathrm{e}^{u_{N+1+j}}}{K'(\lambda_j)}} \qquad \psi_{2j} = \sqrt{\frac{2R(\lambda_j)\,\mathrm{e}^{-u_{N+1+j}}}{K'(\lambda_j)}} \qquad j = 1, \ldots, N. \tag{2.51b}$$

The separated equations (2.49) and (2.50*a*) may be integrated to give the completely separated solution for the generating function S of the canonical transformation

$$S(u_1, \ldots, u_{2N+1}) = \sum_{k=1}^{N+1} \int^{u_k} 2\sqrt{P(\lambda)}\,\mathrm{d}\lambda + 2\sum_{k=1}^{N} P_{N+k}u_{N+1+k} \tag{2.52}$$

where $P(\lambda)$ is given by (2.28).

In exactly the same way, one obtains the Jacobi inversion problem for the FDIHS (2.24)

$$\sum_{k=1}^{N+1} \int^{u_k} \frac{1}{\sqrt{P(\lambda)}}\,\mathrm{d}\lambda = \gamma_0 + 2x \tag{2.53a}$$

$$\sum_{k=1}^{N+1} \int^{u_k} \frac{1}{(\lambda - \lambda_i)\sqrt{P(\lambda)}}\,\mathrm{d}\lambda = \gamma_i \qquad i = 1, \ldots, N \tag{2.53b}$$

$$\sum_{k=1}^{N+1} \int^{u_k} \frac{P_{N+i}}{(\lambda - \lambda_i)^2\sqrt{P(\lambda)}}\,\mathrm{d}\lambda + u_{N+1+i} = \gamma_{N+i} \qquad i = 1, \ldots, N \tag{2.53c}$$

the Jacobi inversion problem for the FDIHS (2.26)

$$\sum_{k=1}^{N+1} \int^{u_k} \frac{1}{\sqrt{P(\lambda)}}\,\mathrm{d}\lambda = \gamma_0 \tag{2.54a}$$

$$\sum_{k=1}^{N+1} \int^{u_k} \frac{1}{(\lambda - \lambda_i)\sqrt{P(\lambda)}}\,\mathrm{d}\lambda = \gamma_i + 2t_1 \qquad i = 1, \ldots, N \tag{2.54b}$$

$$\sum_{k=1}^{N+1} \int^{u_k} \frac{P_{N+i}}{(\lambda - \lambda_i)^2\sqrt{P(\lambda)}}\,\mathrm{d}\lambda + u_{N+1+i} = \gamma_{N+i} \qquad i = 1, \ldots, N. \tag{2.54c}$$

Finally, we have the following proposition.

Proposition 4. *The Jacobi inversion problem for the KdV equation (2.6)*

$$\sum_{k=1}^{N+1}\int^{u_k}\frac{1}{\sqrt{P(\lambda)}}\,\mathrm{d}\lambda=\gamma_0+2x \tag{2.55a}$$

$$\sum_{k=1}^{N+1}\int^{u_k}\frac{1}{(\lambda-\lambda_i)\sqrt{P(\lambda)}}\,\mathrm{d}\lambda=\gamma_i+2t_1 \qquad i=1,\ldots,N \tag{2.55b}$$

$$\sum_{k=1}^{N+1}\int^{u_k}\frac{P_{N+i}}{(\lambda-\lambda_i)^2\sqrt{P(\lambda)}}\,\mathrm{d}\lambda+u_{N+1+i}=\gamma_{N+i} \qquad i=1,\ldots,N. \tag{2.55c}$$

If, by using the Jacobi inversion technique [7], u given by (2.51a) can be found in terms of Riemann theta functions by solving (2.55), then u provides the solution of the KdV equation (2.6).

In general, since the nth KdV equation (2.4) is factorized by the x-FDIHS (2.24) and the t_n-FDIHS obtained from (2.9) under (2.24), the above procedure can be applied to find the Jacobi inversion problem for the nth KdV equation (2.4).

2.2.3. The method can be applied to all high-order binary constrained flows (2.8) and (2.9) as well as the whole KdV hierarchy. For any fixed k_0, by introducing the so-called Jacobi–Ostrogradsky coordinates, the high-order binary x-constrained flow (2.8) can be transformed into a x-FDIHS with a degree of freedom $2N+k_0$. Under the x-FDIHS, the binary t_n-constrained flow (2.9) can also be transformed into a t_n-FDIHS. The Lax representation for the x- and t_n-FDIHS can be deduced from the adjoint representation of (2.1) and (2.2) by using the method in [27, 28]. By means of the Lax matrix we can introduce the first $N+k_0$ canonical variables $u_1,\ldots,u_{N+k_0}$ by the set of zeros of $B(\lambda)$ and $v_k=2A(u_k)$, $k=1,\ldots,N+k_0$. Then the additional N canonical separated variables can be defined by

$$v_{N+k_0+j}=2P_{N+j}=\tfrac{1}{2}(\psi_{1j}\phi_{1j}+\psi_{2j}\phi_{2j}) \qquad u_{N+k_0+j}=\ln\frac{\phi_{1j}}{\psi_{2j}} \qquad j=1,\ldots,N.$$

Finally, since the nth KdV equation (2.4) is factorized by the x-FDIHS and the t_n-FDIHS, in exactly the same way we can obtain the Jacobi inversion problem for (2.4). The above scheme can be applied to all soliton equations admitting 2×2 Lax pairs.

3. The separation of variables for the AKNS equations

3.1. Binary constrained flows of the AKNS hierarchy

For the AKNS spectral problem [30]

$$\phi_x=U(u,\lambda)\,\phi \qquad U(u,\lambda)=\begin{pmatrix}-\lambda & q\\ r & \lambda\end{pmatrix} \qquad \phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \qquad u=\begin{pmatrix}q\\ r\end{pmatrix}. \tag{3.1}$$

Take

$$\phi_{t_n}=V^{(n)}(u,\lambda)\,\phi \qquad V^{(n)}(u,\lambda)=\sum_{i=0}^{n}\begin{pmatrix}a_i & b_i\\ c_i & -a_i\end{pmatrix}\lambda^{n-i} \tag{3.2}$$

where

$$a_0=-1 \qquad b_0=c_0=0 \qquad a_1=0 \qquad b_1=q \qquad c_1=r \qquad a_2=\tfrac{1}{2}qr,\ldots$$

$$\begin{pmatrix} c_{k+1} \\ b_{k+1} \end{pmatrix} = L\begin{pmatrix} c_k \\ b_k \end{pmatrix} \qquad a_k=\partial^{-1}(qc_k-rb_k) \qquad k=1,2,\ldots$$
$$L=\tfrac{1}{2}\begin{pmatrix} \partial-2r\partial^{-1}q & 2r\partial^{-1}r \\ -2q\partial^{-1}q & -\partial+2q\partial^{-1}r \end{pmatrix}. \tag{3.3}$$

The AKNS hierarchy associated with (3.1) and (3.2) reads [30]

$$u_{t_n}=\begin{pmatrix} q \\ r \end{pmatrix}_{t_n}=JL^n\begin{pmatrix} r \\ q \end{pmatrix}=J\frac{\delta H_{n+1}}{\delta u} \qquad n=1,2,\ldots \tag{3.4}$$

$$J=\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \qquad H_n=\frac{2a_{n+1}}{n+1} \qquad \begin{pmatrix} c_n \\ b_n \end{pmatrix}=\frac{\delta H_n}{\delta u} \qquad n=1,2,\ldots.$$

For $n=2$ we have

$$\phi_{t_2}=V^{(2)}(u,\lambda)\,\phi \qquad V^{(2)}=\begin{pmatrix} -\lambda^2+\frac{1}{2}qr & q\lambda-\frac{1}{2}q_x \\ r\lambda+\frac{1}{2}r_x & \lambda^2-\frac{1}{2}qr \end{pmatrix} \tag{3.5}$$

and the AKNS equation (3.4) for $n=2$ reads

$$q_{t_2}=-\tfrac{1}{2}q_{xx}+q^2r \qquad r_{t_2}=\tfrac{1}{2}r_{xx}-r^2q. \tag{3.6}$$

The adjoint AKNS spectral problem is of the same form as equation (2.7). We have

$$\frac{\delta\lambda}{\delta u}=\begin{pmatrix} \delta\lambda/\delta q \\ \delta\lambda/\delta r \end{pmatrix}=\mathrm{Tr}\left[\begin{pmatrix} \phi_1\psi_1 & \phi_1\psi_2 \\ \phi_2\psi_1 & \phi_2\psi_2 \end{pmatrix}\frac{\partial U(u,\lambda)}{\partial u}\right]=\begin{pmatrix} \psi_1\phi_2 \\ \psi_2\phi_1 \end{pmatrix} \tag{3.7}$$

which should be read componentwise [26].

The binary x-constrained flows of the AKNS hierarchy (3.4) are defined by [15, 17, 21]

$$\Phi_{1,x}=-\Lambda\Phi_1+q\Phi_2 \qquad \Phi_{2,x}=r\Phi_1+\Lambda\Phi_2 \tag{3.8a}$$

$$\Psi_{1,x}=\Lambda\Psi_1-r\Psi_2 \qquad \Psi_{2,x}=-q\Psi_1-\Lambda\Psi_2 \tag{3.8b}$$

$$\frac{\delta H_{k_0+1}}{\delta u}-\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}=\begin{pmatrix} c_{k_0+1} \\ b_{k_0+1} \end{pmatrix}-\beta\begin{pmatrix} \langle\Psi_1,\Phi_2\rangle \\ \langle\Psi_2,\Phi_1\rangle \end{pmatrix}=0. \tag{3.8c}$$

3.1.1. For $k_0=0$, $\beta=1$ we have

$$\begin{pmatrix} c_1 \\ b_1 \end{pmatrix}=\begin{pmatrix} r \\ q \end{pmatrix}=\begin{pmatrix} \langle\Psi_1,\Phi_2\rangle \\ \langle\Psi_2,\Phi_1\rangle \end{pmatrix}. \tag{3.9}$$

By substituting (3.9) into (3.8a) and (3.8b), one obtains an x-FDHS [15, 17]

$$\Phi_{1x}=\frac{\partial F_1}{\partial\Psi_1} \qquad \Phi_{2x}=\frac{\partial F_1}{\partial\Psi_2} \qquad \Psi_{1x}=-\frac{\partial F_1}{\partial\Phi_1} \qquad \Psi_{2x}=-\frac{\partial F_1}{\partial\Phi_2} \tag{3.10}$$
$$F_1=\langle\Lambda\Psi_2,\Phi_2\rangle-\langle\Lambda\Psi_1,\Phi_1\rangle+\langle\Psi_2,\Phi_1\rangle\langle\Psi_1,\Phi_2\rangle.$$

Under the constraint (3.9) and the FDHS (3.10), the binary t_2-constrained flow obtained from (3.2) with $V^{(2)}$ given by (3.5) and its adjoint equation for N distinct real number λ_j can also be written as a t_2-FDHS

$$\Phi_{1,t_2}=\frac{\partial F_2}{\partial\Psi_1} \qquad \Phi_{2,t_2}=\frac{\partial F_2}{\partial\Psi_2} \qquad \Psi_{1,t_2}=-\frac{\partial F_2}{\partial\Phi_1} \qquad \Psi_{2,t_2}=-\frac{\partial F_2}{\partial\Phi_2}$$
$$F_2=\langle\Lambda^2\Psi_2,\Phi_2\rangle-\langle\Lambda^2\Psi_1,\Phi_1\rangle+\langle\Psi_2,\Phi_1\rangle\langle\Lambda\Psi_1,\Phi_2\rangle \tag{3.11}$$
$$+\langle\Lambda\Psi_2,\Phi_1\rangle\langle\Psi_1,\Phi_2\rangle-\tfrac{1}{2}(\langle\Psi_2,\Phi_2\rangle-\langle\Psi_1,\Phi_1\rangle)\langle\Psi_2,\Phi_1\rangle\langle\Psi_1,\Phi_2\rangle.$$

The Lax representation for the FDHSs (3.10) and (3.11) which can also be deduced from the adjoint representation of (3.1) and (3.2) are presented by (2.13) with the entries of the Lax matrix M given by [21]

$$A(\lambda) = -1 + \frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j}}{\lambda - \lambda_j} \tag{3.12a}$$

$$B(\lambda) = \sum_{j=1}^{N}\frac{\psi_{2j}\phi_{1j}}{\lambda - \lambda_j} \qquad C(\lambda) = \sum_{j=1}^{N}\frac{\psi_{1j}\phi_{2j}}{\lambda - \lambda_j}. \tag{3.12b}$$

A straightforward calculation yields

$$A^2(\lambda) + B(\lambda)\,C(\lambda) \equiv P(\lambda) = 1 + \sum_{j=1}^{N}\left[\frac{P_j}{\lambda - \lambda_j} + \frac{P_{N+j}^2}{(\lambda - \lambda_j)^2}\right] \tag{3.13}$$

where $P_1, \ldots, P_{2N}$ are independent integrals of motion for the FDHSs (3.10) and (3.11)

$$P_j = \psi_{2j}\phi_{2j} - \psi_{1j}\phi_{1j} + \frac{1}{2}\sum_{k\neq j}\frac{1}{\lambda_j - \lambda_k}$$
$$\times[(\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j})(\psi_{1k}\phi_{1k} - \psi_{2k}\phi_{2k}) + 4\psi_{1j}\phi_{2j}\psi_{2k}\phi_{1k}] \tag{3.14a}$$

$$P_{N+j} = \tfrac{1}{2}(\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}) \qquad j = 1, \ldots, N. \tag{3.14b}$$

It is easy to verify that

$$F_1 = \sum_{j=1}^{N}\left(\lambda_j P_j + P_{N+j}^2\right) - \tfrac{1}{4}\left(\sum_{j=1}^{N} P_j\right)^2 \tag{3.15a}$$

$$F_2 = \sum_{j=1}^{N}\left(\lambda_j^2 P_j + 2\lambda_j P_{N+j}^2\right) - \tfrac{1}{2}\left(\sum_{j=1}^{N} P_j\right)\sum_{j=1}^{N}\left(\lambda_j P_j + P_{N+j}^2\right) + \tfrac{1}{8}\left(\sum_{j=1}^{N} P_j\right)^3. \tag{3.15b}$$

Similarly, it is easy to show that $P_1, \ldots, P_{2N}$ are in involution, equations (3.10) and (3.11) are FDIHSs. The AKNS equation (3.6) is factorized by the x-FDIHS (3.10) and the t_2-FDIHS (3.11), namely, if $(\Psi_1, \Psi_2, \Phi_1, \Phi_2)$ satisfies the FDIHSs (3.10) and (3.11) simultaneously, then (q, r) given by (3.9) solves the AKNS equation (3.6). In general, the factorization of the nth AKNS equations (3.4) will be presented at the end of section 3.2.

3.1.2. For $k_0 = 1, \beta = \frac{1}{2}$, equation (3.8*c*) yields

$$r_x = \langle\Psi_1, \Phi_2\rangle \qquad q_x = -\langle\Psi_2, \Phi_1\rangle. \tag{3.16}$$

Equations (3.8*a*), (3.8*b*) and (3.16) can be written as a x-FDIHS

$$\Phi_{ix} = \frac{\partial F_1}{\partial \Psi_1} \qquad r_x = \frac{\partial F_1}{\partial q} \qquad \Psi_{ix} = -\frac{\partial F_1}{\partial \Phi_i} \qquad q_x = -\frac{\partial F_1}{\partial r} \qquad i = 1, 2$$
$$F_1 = \langle\Lambda\Psi_2, \Phi_2\rangle - \langle\Lambda\Psi_1, \Phi_1\rangle + r\langle\Psi_2, \Phi_1\rangle + q\langle\Psi_1, \Phi_2\rangle. \tag{3.17}$$

Under the constraint (3.16) and the FDIHS (3.17), $V^{(2)}$ becomes

$$\tilde{V}^{(2)} = \begin{pmatrix} -\lambda^2 + \frac{1}{2}qr & q\lambda + \frac{1}{2}\langle\Psi_2, \Phi_1\rangle \\ r\lambda + \frac{1}{2}\langle\Psi_1, \Phi_2\rangle & \lambda^2 - \frac{1}{2}qr \end{pmatrix}. \tag{3.18}$$

Then under the constraint (3.16) and the FDIHS (3.17), the binary t_2-constrained flow consisting of replicas (3.5) and its adjoint system for N distinct real numbers λ_j as well as (3.6) can also be written as a t_2-FDIHS

$$\Phi_{i,t_2} = \frac{\partial F_2}{\partial \Psi_i} \qquad r_{t_2} = \frac{\partial F_2}{\partial q} \qquad \Psi_{i,t_2} = -\frac{\partial F_2}{\partial \Phi_i} \qquad q_{t_2} = -\frac{\partial F_2}{\partial r} \qquad i = 1, 2$$
$$F_2 = \langle \Lambda^2 \Psi_2, \Phi_2 \rangle - \langle \Lambda^2 \Psi_1, \Phi_1 \rangle + q \langle \Lambda \Psi_1, \Phi_2 \rangle + r \langle \Lambda \Psi_2, \Phi_1 \rangle$$
$$\qquad - \tfrac{1}{2} qr (\langle \Psi_2, \Phi_2 \rangle - \langle \Psi_1, \Phi_1 \rangle) + \tfrac{1}{2} \langle \Psi_2, \Phi_1 \rangle \langle \Psi_1, \Phi_2 \rangle - \tfrac{1}{2} q^2 r^2. \tag{3.19}$$

The Lax matrix M for FDIHS (3.17) and (3.19) is given by

$$A(\lambda) = -\lambda + \frac{1}{4} \sum_{j=1}^{N} \frac{\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j}}{\lambda - \lambda_j} \tag{3.20a}$$

$$B(\lambda) = q + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{2j}\phi_{1j}}{\lambda - \lambda_j} \qquad C(\lambda) = r + \frac{1}{2} \sum_{j=1}^{N} \frac{\psi_{1j}\phi_{2j}}{\lambda - \lambda_j}. \tag{3.20b}$$

A straightforward calculation yields

$$A^2(\lambda) + B(\lambda)\, C(\lambda) \equiv P(\lambda) = \lambda^2 + P_0 + \sum_{j=1}^{N} \left[\frac{P_j}{\lambda - \lambda_j} + \frac{P_{N+j}^2}{(\lambda - \lambda_j)^2} \right] \tag{3.21}$$

where $P_0, \ldots, P_{2N}$ are independent integrals of motion in involution for the FDIHSs (3.17) and (3.19)

$$P_0 = \tfrac{1}{2}(\langle \Psi_2, \Phi_2 \rangle - \langle \Psi_1, \Phi_1 \rangle) + qr$$

$$P_j = \tfrac{1}{2}[\lambda_j \psi_{2j}\phi_{2j} - \lambda_j \psi_{1j}\phi_{1j} + q\psi_{1j}\phi_{2j} + r\psi_{2j}\phi_{1j}]$$
$$\qquad + \frac{1}{8} \sum_{k \neq j} \frac{1}{\lambda_j - \lambda_k} [(\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j})(\psi_{1k}\phi_{1k} - \psi_{2k}\phi_{2k}) + 4\psi_{1j}\phi_{2j}\psi_{2k}\phi_{1k}]$$

$$P_{N+j} = \tfrac{1}{4}(\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}) \qquad j = 1, \ldots, N.$$

It is easy to verify that

$$F_1 = 2 \sum_{j=1}^{N} P_j \qquad F_2 = 2 \sum_{j=1}^{N} \left(\lambda_j P_j + P_{N+j}^2 \right) - \tfrac{1}{2} P_0^2. \tag{3.22}$$

It is easy to show that $P_1, \ldots, P_{2N}$ are in involution, equations (3.17) and (3.18) are FDIHSs and commute with each other. The AKNS equation (3.6) is factorized by the x-FDIHS (3.17) and the t_2-FDIHS (3.19), namely, if $(\Psi_1, \Psi_2, q, \Phi_1, \Phi_2, r)$ satisfies the FDIHSs (3.17) and (3.19) simultaneously, then (q, r) solves the AKNS equation (3.6).

3.2. The separation of variables for the AKNS equations

3.2.1. For $k_0 = 0$ case we present the Jacobi inversion problem for (3.10) and (3.11) as well as for (3.6). With respect to the standard Poisson bracket, $A(\lambda)$ and $B(\lambda)$ given by (3.12) satisfy

$$\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = 0 \qquad \{A(\lambda), B(\mu)\} = \frac{1}{\lambda - \mu}[B(\mu) - B(\lambda)]. \tag{3.23}$$

In contrast with $B(\lambda)$ for the constrained KdV flows, $B(\lambda)$ given by (3.12*b*) has only $N-1$ zeros, one has to define the canonical variables $u_k, v_k, k = 1, \ldots, N$, in a slightly different way:

$$B(\lambda) = \sum_{j=1}^{N} \frac{\psi_{2j}\phi_{1j}}{\lambda - \lambda_j} = e^{u_N} \frac{R(\lambda)}{K(\lambda)} \qquad R(\lambda) = \prod_{k=1}^{N-1}(\lambda - u_k) \qquad K(\lambda) = \prod_{k=1}^{N}(\lambda - \lambda_k) \tag{3.24a}$$

$$v_k = A(u_k) \qquad k = 1, \ldots, N-1 \qquad v_N = \tfrac{1}{2}(\langle\Psi_1, \Phi_1\rangle - \langle\Psi_2, \Phi_2\rangle). \tag{3.24b}$$

Equation (3.24*a*) yields

$$u_N = \ln\langle\Psi_2, \Phi_1\rangle. \tag{3.24c}$$

Then it is easy to verify that

$$\{u_N, B(\mu)\} = \{v_N, A(\mu)\} = 0 \qquad \{v_N, u_N\} = 1 \tag{3.25a}$$

$$\{u_N, A(\mu)\} = -\frac{B(\mu)}{\langle\Psi_2, \Phi_1\rangle} \qquad \{v_N, B(\mu)\} = B(\mu). \tag{3.25b}$$

The commutator relations (3.23) and (3.25) guarantee that $u_1, \ldots, u_N, v_1, \ldots, v_N$ satisfy the canonical conditions (1.1). Similarly, we define

$$v_{N+j} = P_{N+j} \qquad u_{N+j} = \ln\frac{\phi_{1j}}{\psi_{2j}} \qquad j = 1, \ldots, N. \tag{3.26}$$

In the same way we can show the following proposition.

Proposition 5. *Assume that* $\lambda_j, \phi_{ij}, \psi_{ij} \in \mathbb{R}, i = 1, 2, j = 1, \ldots, N$. *Introduce the separated variables* $u_1, \ldots, u_{2N}$ *and* $v_1, \ldots, v_{2N}$ *by (3.24) and (3.26). If* $u_1, \ldots, u_{N-1}$, *are single zeros of* $B(\lambda)$, *then* $v_1, \ldots, v_{2N}$ *and* $u_1, \ldots, u_{2N}$ *are canonically conjugated, i.e. they satisfy (1.1).*

It follows from (3.24) that

$$q = e^{u_N}$$
$$\psi_{2j}\phi_{1j} = e^{u_N}\frac{R(\lambda_j)}{K'(\lambda_j)} \qquad \frac{\phi_{1j}}{\psi_{2j}} = e^{u_{N+j}} \qquad j = 1, \ldots, N \tag{3.27}$$

or

$$\phi_{1j} = \sqrt{\frac{e^{u_N + u_{N+j}} R(\lambda_j)}{K'(\lambda_j)}} \qquad \psi_{2j} = \sqrt{\frac{e^{u_N - u_{N+j}} R(\lambda_j)}{K'(\lambda_j)}} \qquad j = 1, \ldots, N. \tag{3.28}$$

It is easy to see from (3.13) that

$$v_N = \tfrac{1}{2}(\langle\Psi_1, \Phi_1\rangle - \langle\Psi_2, \Phi_2\rangle) = -\tfrac{1}{2}\sum_{i=1}^{N} P_i. \tag{3.29}$$

Then the separated equations obtained by substituting u_k into (3.13) and using (3.24) and the separated equations (3.26) and (3.29) may be integrated to give the generating function of the canonical transformation

$$S(u_1, \ldots, u_{2N}) = \sum_{k=1}^{N-1}\int^{u_k}\sqrt{P(\lambda)}\,d\lambda - \tfrac{1}{2}\sum_{i=1}^{N} P_i u_N + \sum_{i=1}^{N} P_{N+i}u_{N+i}. \tag{3.30}$$

The linearizing coordinates are then

$$Q_i = \frac{\partial S}{\partial P_i} = \frac{1}{2}\sum_{k=1}^{N-1}\int^{u_k}\frac{1}{(\lambda-\lambda_i)\sqrt{P(\lambda)}}\,\mathrm{d}\lambda - \frac{1}{2}u_N \qquad i=1,\ldots,N \tag{3.31a}$$

$$Q_{N+i} = \frac{\partial S}{\partial P_{N+i}} = \sum_{k=1}^{N-1}\int^{u_k}\frac{P_{N+i}}{(\lambda-\lambda_i)^2\sqrt{P(\lambda)}}\,\mathrm{d}\lambda + u_{N+i} \qquad i=1,\ldots,N. \tag{3.31b}$$

By using (3.15a), the linear flow induced by the FDIHS (3.10) together with equations (3.31) leads to the Jacobi inversion problem for the FDIHS (3.10)

$$\sum_{k=1}^{N-1}\int^{u_k}\frac{1}{(\lambda-\lambda_i)\sqrt{P(\lambda)}}\,\mathrm{d}\lambda - u_N = \gamma_i + \left(2\lambda_i - \sum_{k=1}^{N}P_k\right)x \qquad i=1,\ldots,N \tag{3.32a}$$

$$\sum_{k=1}^{N-1}\int^{u_k}\frac{P_{N+i}}{(\lambda-\lambda_i)^2\sqrt{P(\lambda)}}\,\mathrm{d}\lambda + u_{N+i} = \gamma_{N+i} + 2P_{N+i}x \qquad i=1,\ldots,N. \tag{3.32b}$$

By using (3.15b), the linear flow induced by the FDIHS (3.11) and equations (3.31) result in the Jacobi inversion problem for the FDIHS (3.11)

$$\sum_{k=1}^{N-1}\int^{u_k}\frac{1}{(\lambda-\lambda_i)\sqrt{P(\lambda)}}\,\mathrm{d}\lambda - u_N = \bar{\gamma}_i + \left[2\lambda_i^2 - \sum_{k=1}^{N}(\lambda_k P_k + \lambda_i P_k + P_{N+k}^2) + \frac{3}{4}\left(\sum_{k=1}^{N}P_k\right)^2\right]t_2$$
$$i=1,\ldots,N \tag{3.33a}$$

$$\sum_{k=1}^{N-1}\int^{u_k}\frac{P_{N+i}}{(\lambda-\lambda_i)^2\sqrt{P(\lambda)}}\,\mathrm{d}\lambda + u_{N+i} = \bar{\gamma}_{N+i} + P_{N+i}\left(4\lambda_i - \sum_{k=1}^{N}P_k\right)t_2$$
$$i=1,\ldots,N. \tag{3.33b}$$

Then, since the AKNS equations (3.6) are factorized by the FDIHSs (3.10) and (3.11), combining equations (3.32) with equations (3.33) gives rise to the Jacobi inversion problem for the AKNS equations (3.6)

$$\sum_{k=1}^{N-1}\int^{u_k}\frac{1}{(\lambda-\lambda_i)\sqrt{P(\lambda)}}\,\mathrm{d}\lambda - u_N = \gamma_i + \left(2\lambda_i - \sum_{k=1}^{N}P_k\right)x$$
$$+\left[2\lambda_i^2 - \sum_{k=1}^{N}\left(\lambda_k P_k + \lambda_i P_k + P_{N+k}^2\right) + \frac{3}{4}\left(\sum_{k=1}^{N}P_k\right)^2\right]t_2 \tag{3.34a}$$

$$\sum_{k=1}^{N}\int^{u_k}\frac{P_{N+i}}{(\lambda-\lambda_i)^2\sqrt{P(\lambda)}}\,\mathrm{d}\lambda + u_{N+i} = \gamma_{N+i} + 2P_{N+i}x + P_{N+i}\left(4\lambda_i - \sum_{k=1}^{N}P_k\right)t_2$$
$$i=1,\ldots,N. \tag{3.34b}$$

If ϕ_{1j}, ψ_{2j}, q defined by (3.27) and (3.28) can be solved from (3.34) by using the Jacobi inversion technique, then ϕ_{2j}, ψ_{1j} can be obtained from the first equation and the last equation in (3.10) by an algebraic calculation, respectively. Finally, q and $r = \langle\Psi_1, \Phi_2\rangle$ provide the solution to the AKNS equations (3.6).

Comparing (2.20) with (3.13), one obtains

$$\tilde{F}_0 = 1 \qquad \tilde{F}_k = \sum_{j=1}^{N}\left[\lambda_j^{k-1}P_j + (k-1)\lambda_j^{k-2}P_{N+j}^2\right] \qquad k=1,2,\ldots \tag{3.35}$$

where $\tilde{F}_k$, $k = 1, 2, \ldots,$ are also integrals of motion for both the FDIHS (3.10) and the t_n-binary constrained flow. The nth AKNS equations (3.4) are factorized by the x-FDIHS (3.10) and the t_n-FDIHS with the Hamiltonian F_n given by

$$F_n = 2\sum_{m=0}^{n}\left(-\tfrac{1}{2}\right)^m \frac{\alpha_m}{m+1} \sum_{l_1+\cdots+l_{m+1}=n+1} \tilde{F}_{l_1}\ldots\tilde{F}_{l_{m+1}} \qquad (3.36)$$

where $l_1 \geqslant 1, \ldots, l_{m+1} \geqslant 1$, α_m are given by (2.22*c*). We have the following proposition:

Proposition 6. *The Jacobi inversion problem for the nth AKNS equations (3.4) is*

$$\sum_{k=1}^{N-1}\int^{u_k} \frac{1}{(\lambda-\lambda_i)\sqrt{P(\lambda)}}\,\mathrm{d}\lambda - u_N = \gamma_i + \left(2\lambda_i - \sum_{k=1}^{N} P_k\right)x$$
$$+2t_n\sum_{m=0}^{n}(-\tfrac{1}{2})^m\alpha_m \sum_{l_1+\cdots+l_{m+1}=n+1} \lambda_i^{l_{m+1}-1}\tilde{F}_{l_1}\ldots\tilde{F}_{l_m} \qquad i = 1, \ldots, N$$

$$\sum_{k=1}^{N}\int^{u_k} \frac{P_{N+i}}{(\lambda-\lambda_i)^2\sqrt{P(\lambda)}}\,\mathrm{d}\lambda + u_{N+i} = \gamma_{N+i} + 2P_{N+i}x + 4t_n\sum_{m=0}^{n}(-\tfrac{1}{2})^m\alpha_m$$
$$\times \sum_{l_1+\cdots+l_{m+1}=n+1}(l_{m+1}-1)\lambda_i^{l_{m+1}-2}P_{N+i}\tilde{F}_{l_1}\ldots\tilde{F}_{l_m} \qquad i = 1, \ldots, N$$

where $l_1 \geqslant 1, \ldots, l_{m+1} \geqslant 1$, and $\tilde{F}_{l_1}, \ldots \tilde{F}_{l_m}$, are given by (3.35).

3.2.2. For the $k_0 = 1$ case with respect to the standard Poisson bracket, $A(\lambda)$ and $B(\lambda)$ given by (3.20) also satisfy the commutator relation (2.31). One defines the first $N+1$ pair of canonical variables u_k, v_k, $k = 1, \ldots, N+1$, in the following way:

$$B(\lambda) = q + \frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{2j}\phi_{1j}}{\lambda-\lambda_j} = \mathrm{e}^{u_{N+1}}\frac{R(\lambda)}{K(\lambda)} \qquad (3.37a)$$

with

$$R(\lambda) = \prod_{k=1}^{N}(\lambda - u_k) \qquad K(\lambda) = \prod_{k=1}^{N}(\lambda-\lambda_k)$$

and

$$v_k = 2A(u_k) \qquad k = 1, \ldots, N \qquad (3.37b)$$

$$v_{N+1} = P_0 = qr - \tfrac{1}{2}(\langle\Psi_1, \Phi_1\rangle - \langle\Psi_2, \Phi_2\rangle). \qquad (3.37c)$$

Equation (3.24*a*) yields

$$u_{N+1} = \ln q. \qquad (3.37d)$$

Then it is easy to verify that

$$\{u_{N+1}, B(\mu)\} = \{v_{N+1}, A(\mu)\} = 0 \qquad \{v_{N+1}, u_{N+1}\} = 1$$
$$\{u_{N+1}, A(\mu)\} = 0 \qquad \{v_{N+1}, B(\mu)\} = B(\mu). \qquad (3.38)$$

Similarly, we define

$$v_{N+1+j} = 2P_{N+j} \qquad j = 1, \ldots, N \qquad (3.39a)$$

$$u_{N+1+j} = \ln\frac{\phi_{1j}}{\psi_{2j}} \qquad j = 1, \ldots, N. \qquad (3.39b)$$

In the same way we can show the following proposition.

Proposition 7. *Assume that* $\lambda_j, \phi_{ij}, \psi_{ij} \in \mathbb{R}, i = 1, 2, j = 1, \ldots, N$. *Introduce the separated variables* $u_1, \ldots, u_{2N+1}$ *and* $v_1, \ldots, v_{2N+1}$ *by (3.37) and (3.39). If* $u_1, \ldots, u_N$, *are single zeros of* $B(\lambda)$, *then* $v_1, \ldots, v_{2N+1}$ *and* $u_1, \ldots, u_{2N+1}$ *are canonically conjugated, i.e. they satisfy (1.1).*

It follows from (3.37) that

$$q = e^{u_{N+1}} \tag{3.40a}$$

$$\phi_{1j} = \sqrt{\frac{2e^{u_{N+1}+u_{N+1+j}} R(\lambda_j)}{K'(\lambda_j)}} \qquad \psi_{2j} = \sqrt{\frac{2e^{u_{N+1}-u_{N+1+j}} R(\lambda_j)}{K'(\lambda_j)}} \qquad j = 1, \ldots, N. \tag{3.40b}$$

The first N separated equations can be found by substituting u_k into (3.21) and using (3.37*b*), the last $N + 1$ separated equations are given by (3.37*c*) and (3.39*a*). They may be integrated to give

$$S(u_1, \ldots, u_{2N+1}) = \sum_{k=1}^{N} \left(2 \int^{u_k} \sqrt{P(\lambda)}\, d\lambda + 2P_{N+k} u_{N+1+k} \right) + P_0 u_{N+1} \tag{3.41}$$

with $P(\lambda)$ given by (3.21). Then the Jacobi inversion problem for the FDIHS (3.17) is

$$\begin{aligned}
&\sum_{k=1}^{N} \int^{u_k} \frac{1}{\sqrt{P(\lambda)}}\, d\lambda + u_{N+1} = \gamma_0 \\
&\sum_{k=1}^{N} \int^{u_k} \frac{1}{(\lambda - \lambda_i)\sqrt{P(\lambda)}}\, d\lambda = \gamma_i + 2x \qquad i = 1, \ldots, N \\
&\sum_{k=1}^{N} \int^{u_k} \frac{P_{N+i}}{(\lambda - \lambda_i)^2\sqrt{P(\lambda)}}\, d\lambda + u_{N+1+i} = \gamma_{N+i} \qquad i = 1, \ldots, N.
\end{aligned} \tag{3.42}$$

The Jacobi inversion problem for the FDIHS (3.19) is

$$\begin{aligned}
&\sum_{k=1}^{N} \int^{u_k} \frac{1}{\sqrt{P(\lambda)}}\, d\lambda + u_{N+1} = \gamma_0 - P_0 t_2 \\
&\sum_{k=1}^{N} \int^{u_k} \frac{1}{(\lambda - \lambda_i)\sqrt{P(\lambda)}}\, d\lambda = \gamma_i + 2\lambda_i t_2 \qquad i = 1, \ldots, N \\
&\sum_{k=1}^{N} \int^{u_k} \frac{P_{N+i}}{(\lambda - \lambda_i)^2\sqrt{P(\lambda)}}\, d\lambda + u_{N+1+i} = \gamma_{N+i} + 2P_{N+i} t_2 \qquad i = 1, \ldots, N.
\end{aligned} \tag{3.43}$$

Finally, we have

Proposition 8. *The Jacobi inversion problem for the AKNS equation (3.6) is*

$$\begin{aligned}
&\sum_{k=1}^{N} \int^{u_k} \frac{1}{\sqrt{P(\lambda)}}\, d\lambda + u_{N+1} = \gamma_0 - P_0 t_2 \\
&\sum_{k=1}^{N} \int^{u_k} \frac{1}{(\lambda - \lambda_i)\sqrt{P(\lambda)}}\, d\lambda = \gamma_i + 2(x + \lambda_i t_2) \qquad i = 1, \ldots, N \\
&\sum_{k=1}^{N} \int^{u_k} \frac{P_{N+i}}{(\lambda - \lambda_i)^2\sqrt{P(\lambda)}}\, d\lambda + u_{N+1+i} = \gamma_{N+i} + 2P_{N+i} t_2 \qquad i = 1, \ldots, N.
\end{aligned} \tag{3.44}$$

If ϕ_{1j}, ψ_{2j}, q defined by (3.40) can be solved from (3.44) by using the Jacobi inversion technique, then ϕ_{2j}, ψ_{1j} and r can be obtained from the equations in (3.17) by an algebraic calculation, respectively. Finally, (q, r) provides the solution to the AKNS equations (3.6).

3.2.3. The above procedure can be applied to all high-order binary constrained flows (3.8) and whole AKNS hierarchy (3.4).

4. The separation of variables for the Kaup–Newell equations

4.1. Binary constrained flows of the Kaup–Newell hierarchy

For the Kaup–Newell spectral problem [31]

$$\phi_x = U(u,\lambda)\,\phi \qquad U(u,\lambda) = \begin{pmatrix} -\lambda^2 & q\lambda \\ r\lambda & \lambda^2 \end{pmatrix} \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad u = \begin{pmatrix} q \\ r \end{pmatrix} \tag{4.1}$$

take

$$\phi_{t_n} = V^{(n)}(u,\lambda)\,\phi \qquad V^{(n)}(u,\lambda) = \sum_{i=0}^{n-1} \begin{pmatrix} a_{2i}\lambda^{2n-2i} & b_{2i+1}\lambda^{2n-2i-1} \\ c_{2i+1}\lambda^{2n-2i-1} & -a_{2i}\lambda^{2n-2i} \end{pmatrix} \tag{4.2}$$

where

$$a_0 = 1 \qquad a_2 = -\tfrac{1}{2}qr \qquad b_1 = -q$$
$$c_1 = -r \qquad b_3 = \tfrac{1}{2}(q^2r + q_x) \qquad c_3 = \tfrac{1}{2}(qr^2 - r_x), \ldots$$

and in general $a_{2k+1} = b_{2k} = c_{2k} = 0$

$$\begin{pmatrix} c_{2k+1} \\ b_{2k+1} \end{pmatrix} = L \begin{pmatrix} c_{2k-1} \\ b_{2k-1} \end{pmatrix} \qquad a_{2k} = \tfrac{1}{2}\partial^{-1}(qc_{2k-1,x} + rb_{2k-1,x}) \qquad k = 1, 2, \ldots$$
$$L = \tfrac{1}{2}\begin{pmatrix} \partial - r\partial^{-1}q\partial & -r\partial^{-1}r\partial \\ -q\partial^{-1}q\partial & -\partial - q\partial^{-1}r\partial \end{pmatrix}. \tag{4.3}$$

Then the compatibility condition of equations (4.1) and (4.2) gives rise to the Kaup–Newell hierarchy [31]

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J\begin{pmatrix} c_{2n-1} \\ b_{2n-1} \end{pmatrix} = J\frac{\delta H_{2n-2}}{\delta u} \qquad n = 1, 2, \ldots \tag{4.4}$$

where the Hamiltonian H_n and the Hamiltonian operator J are given by

$$J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \qquad H_{2n} = \frac{4a_{2n+2} - rc_{2n+1} - qb_{2n+1}}{2n} \qquad \begin{pmatrix} c_{2n+1} \\ b_{2n+1} \end{pmatrix} = \frac{\delta H_{2n}}{\delta u}.$$

For $n = 2$ we have

$$\phi_{t_2} = V^{(2)}(u,\lambda)\,\phi \qquad V^{(2)} = \begin{pmatrix} \lambda^4 - \frac{1}{2}qr\lambda^2 & -q\lambda^3 + \frac{1}{2}(q^2r + q_x)\lambda \\ -r\lambda^3 + \frac{1}{2}(qr^2 - r_x)\lambda & -\lambda^4 + \frac{1}{2}qr\lambda^2 \end{pmatrix} \tag{4.5}$$

and the coupled derivative nonlinear Schrödinger (CDNS) equations obtained from equation (4.4) for $n = 2$ read

$$q_{t_2} = \tfrac{1}{2}q_{xx} + \tfrac{1}{2}(q^2r)_x \qquad r_{t_2} = -\tfrac{1}{2}r_{xx} + \tfrac{1}{2}(r^2q)_x. \tag{4.6}$$

The adjoint Kaup–Newell spectral problem is equation (2.7) with U given by (4.1). We have [26]

$$\frac{\delta\lambda}{\delta u} = \begin{pmatrix} \delta\lambda/\delta q \\ \delta\lambda/\delta r \end{pmatrix} = \mathrm{Tr}\left[\begin{pmatrix} \phi_1\psi_1 & \phi_1\psi_2 \\ \phi_2\psi_1 & \phi_2\psi_2 \end{pmatrix}\frac{\partial U(u,\lambda)}{\partial u}\right] = \begin{pmatrix} \lambda\psi_1\phi_2 \\ \lambda\psi_2\phi_1 \end{pmatrix}. \tag{4.7}$$

The binary x-constrained flows of the Kaup–Newell hierarchy (4.4) are defined by

$$\Phi_{1,x} = -\Lambda^2\Phi_1 + q\Lambda\Phi_2 \qquad \Phi_{2,x} = r\Lambda\Phi_1 + \Lambda^2\Phi_2 \tag{4.8a}$$

$$\Psi_{1,x} = \Lambda^2\Psi_1 - r\Lambda\Psi_2 \qquad \Psi_{2,x} = -q\Lambda\Psi_1 - \Lambda^2\Psi_2 \tag{4.8b}$$

$$\frac{\delta H_{k_0}}{\delta u} - \sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} = \begin{pmatrix} c_{2k_0+1} \\ b_{2k_0+1} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \langle\Lambda\Psi_1, \Phi_2\rangle \\ \langle\Lambda\Psi_2, \Phi_1\rangle \end{pmatrix} = 0. \tag{4.8c}$$

For $k_0 = 0$, we have

$$\begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = -\begin{pmatrix} r \\ q \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \langle\Lambda\Psi_1, \Phi_2\rangle \\ \langle\Lambda\Psi_2, \Phi_1\rangle \end{pmatrix}. \tag{4.9}$$

By substituting (4.9) into (4.8a) and (4.8b), the first binary x-constrained flow becomes a FDHS

$$\Phi_{1x} = \frac{\partial F_1}{\partial\Psi_1} \qquad \Phi_{2x} = \frac{\partial F_1}{\partial\Psi_2} \qquad \Psi_{1x} = -\frac{\partial F_1}{\partial\Phi_1} \qquad \Psi_{2x} = -\frac{\partial F_1}{\partial\Phi_2} \tag{4.10}$$

with the Hamiltonian

$$F_1 = \langle\Lambda^2\Psi_2, \Phi_2\rangle - \langle\Lambda^2\Psi_1, \Phi_1\rangle - \tfrac{1}{2}\langle\Lambda\Psi_2, \Phi_1\rangle\langle\Lambda\Psi_1, \Phi_2\rangle.$$

Under the constraint (4.9) and the FDHS (4.10), the binary t_2-constrained flow obtained from (4.5) and its adjoint equation for N distinct real numbers λ_j can also be written as a FDHS

$$\Phi_{1,t_2} = \frac{\partial F_2}{\partial\Psi_1} \qquad \Phi_{2,t_2} = \frac{\partial F_2}{\partial\Psi_2} \qquad \Psi_{1,t_2} = -\frac{\partial F_2}{\partial\Phi_1} \qquad \Psi_{2,t_2} = -\frac{\partial F_2}{\partial\Phi_2} \tag{4.11}$$

with the Hamiltonian

$$\begin{aligned} F_2 = {} & -\langle\Lambda^4\Psi_2, \Phi_2\rangle + \langle\Lambda^4\Psi_1, \Phi_1\rangle + \tfrac{1}{2}\langle\Lambda\Psi_2, \Phi_1\rangle\langle\Lambda^3\Psi_1, \Phi_2\rangle \\ & +\tfrac{1}{2}\langle\Lambda^3\Psi_2, \Phi_1\rangle\langle\Lambda\Psi_1, \Phi_2\rangle - \tfrac{1}{32}\langle\Lambda\Psi_2, \Phi_1\rangle^2\langle\Lambda\Psi_1, \Phi_2\rangle^2 \\ & +\tfrac{1}{8}(\langle\Lambda^2\Psi_2, \Phi_2\rangle - \langle\Lambda^2\Psi_1, \Phi_1\rangle)\langle\Lambda\Psi_2, \Phi_1\rangle\langle\Lambda\Psi_1, \Phi_2\rangle. \end{aligned}$$

The Lax representation for the FDHSs (4.10) and (4.11) are presented by (2.13) with the entries of the Lax matrix M given by

$$A(\lambda) = 1 + \frac{1}{4}\sum_{j=1}^{N}\frac{\lambda_j^2(\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j})}{\lambda^2 - \lambda_j^2} \tag{4.12a}$$

$$B(\lambda) = \frac{1}{2}\lambda\sum_{j=1}^{N}\frac{\lambda_j\psi_{2j}\phi_{1j}}{\lambda^2 - \lambda_j^2} \qquad C(\lambda) = \frac{1}{2}\lambda\sum_{j=1}^{N}\frac{\lambda_j\psi_{1j}\phi_{2j}}{\lambda^2 - \lambda_j^2}. \tag{4.12b}$$

A straightforward calculation yields

$$A^2(\lambda) + B(\lambda)\,C(\lambda) \equiv P(\lambda) = 1 + \sum_{j=1}^{N}\left[\frac{P_j}{\lambda^2 - \lambda_j^2} + \frac{\lambda_j^4 P_{N+j}^2}{(\lambda^2 - \lambda_j^2)^2}\right] \tag{4.13}$$

where P_j, $j = 1, \ldots, 2N$, are $2N$ independent integrals of motion for the FDHSs (4.10) and (4.11)

$$\begin{aligned} P_j = {} & -\tfrac{1}{2}\lambda_j^2(\psi_{2j}\phi_{2j} - \psi_{1j}\phi_{1j}) + \tfrac{1}{8}\langle\Lambda\Psi_2, \Phi_1\rangle\lambda_j\psi_{1j}\phi_{2j} + \tfrac{1}{8}\langle\Lambda\Psi_1, \Phi_2\rangle\lambda_j\psi_{2j}\phi_{1j} \\ & +\frac{1}{8}\sum_{k\neq j}\frac{1}{\lambda_j^2 - \lambda_k^2}\big[\lambda_j^2\lambda_k^2(\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j})(\psi_{1k}\phi_{1k} - \psi_{2k}\phi_{2k}) \\ & +2\lambda_j\lambda_k\big(\lambda_j^2 + \lambda_k^2\big)\psi_{1j}\phi_{2j}\psi_{2k}\phi_{1k}\big] \qquad j = 1, \ldots, N \end{aligned} \tag{4.14a}$$

$$P_{N+j} = \tfrac{1}{4}(\psi_{1j}\phi_{1j} + \psi_{2j}\phi_{2j}) \qquad j = 1, \ldots, N. \tag{4.14b}$$

It is easy to verify that

$$F_1 = -2\sum_{j=1}^{N} P_j \qquad F_2 = 2\sum_{j=1}^{N}\big(\lambda_j^2 P_j + \lambda_j^4 P_{N+j}^2\big) - \tfrac{1}{2}\Bigg(\sum_{j=1}^{N} P_j\Bigg)^2 \tag{4.15a}$$

$$\langle\Psi_2, \Phi_2\rangle + \langle\Psi_1, \Phi_1\rangle = 4\sum_{j=1}^{N} P_{N+j}. \tag{4.15b}$$

By inserting $\lambda = 0$, equation (4.13) leads to

$$1 + \tfrac{1}{4}(\langle\Psi_2, \Phi_2\rangle - \langle\Psi_1, \Phi_1\rangle) = \sqrt{P(0)} = \sqrt{1 + \sum_{j=1}^{N}\big[-P_j\lambda_j^{-2} + P_{N+j}^2\big]}. \tag{4.16}$$

With respect to the standard Poisson bracket it is found that

$$\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} \tag{4.17a}$$

$$\{A(\lambda), B(\mu)\} = \frac{\mu}{2(\lambda^2 - \mu^2)}[\mu B(\mu) - \lambda B(\lambda)]. \tag{4.17b}$$

Then $\{A^2(\lambda) + B(\lambda)\,C(\lambda), A^2(\mu) + B(\mu)\,C(\mu)\} = 0$ implies that P_j, $j = 1, \ldots, 2N$, are in involution. The CDNS equations (4.6) are factorized by the x-FDIHS (4.10) and the t_2-FDIHS (4.11), namely, if $(\Psi_1, \Psi_2, \Phi_1, \Phi_2)$ satisfies the FDIHSs (4.10) and (4.11) simultaneously, then (q, r) given by (4.9) solves the CDNS equations (4.6). The factorization of the nth Kaup–Newell equations (4.4) will be presented in the end of section 4.2.

4.2. The separation of variables for the Kaup–Newell equations

Since the commutator relations (4.17) are quite different from (2.31) and (3.23), we have to modify a little bit of the method presented in sections 2 and 3. Let us denote $\tilde\lambda = \lambda^2$, $\tilde\lambda_j = \lambda_j^2$. The entries of the Lax matrix M given by (4.12) can be rewritten as

$$A(\tilde\lambda) = 1 + \tfrac{1}{4}(\langle\Psi_2, \Phi_2\rangle - \langle\Psi_1, \Phi_1\rangle) + \tfrac{1}{2}\tilde\lambda\bar A(\tilde\lambda) \qquad B(\tilde\lambda) = \tfrac{1}{2}\sqrt{\tilde\lambda}\,\bar B(\tilde\lambda) \tag{4.18a}$$

where

$$\bar A(\tilde\lambda) = \frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}\phi_{1j} - \psi_{2j}\phi_{2j}}{\tilde\lambda - \tilde\lambda_j} \qquad \bar B(\tilde\lambda) = \sum_{j=1}^{N}\frac{\sqrt{\tilde\lambda_j}\,\psi_{2j}\phi_{1j}}{\tilde\lambda - \tilde\lambda_j}. \tag{4.18b}$$

It is easy to see that

$$\{\bar A(\tilde\lambda), \bar A(\tilde\mu)\} = \{\bar B(\tilde\lambda), \bar B(\tilde\mu)\} = 0 \tag{4.19a}$$

$$\{\bar A(\tilde\lambda), \bar B(\tilde\mu)\} = \frac{1}{\tilde\lambda - \tilde\mu}[\bar B(\tilde\mu) - \bar B(\tilde\lambda)]. \tag{4.19b}$$

It follows from (4.16) and (4.18*a*) that

$$A(\tilde\lambda) = \sqrt{1 + \sum_{j=1}^{N}\big[-P_j\tilde\lambda_j^{-1} + P_{N+j}^2\big]} + \tfrac{1}{2}\tilde\lambda\bar A(\tilde\lambda). \tag{4.19c}$$

The commutator relations (4.19) and the generating function of integrals of motion (4.13) enable us to introduce $u_1, \ldots, u_N$ in the following way:

$$\bar B(\tilde\lambda) = \sum_{j=1}^{N}\frac{\sqrt{\tilde\lambda_j}\,\psi_{2j}\phi_{1j}}{\tilde\lambda - \tilde\lambda_j} = e^{u_N}\frac{R(\tilde\lambda)}{K(\tilde\lambda)} \tag{4.20a}$$

with

$$R(\tilde{\lambda}) = \prod_{k=1}^{N-1}(\tilde{\lambda} - u_k) \qquad K(\tilde{\lambda}) = \prod_{k=1}^{N}(\tilde{\lambda} - \tilde{\lambda}_k)$$

and $v_1, \ldots, v_N$ by $\bar{A}(\tilde{\lambda})$:

$$v_k = \bar{A}(u_k) \qquad k = 1, \ldots, N-1 \tag{4.20b}$$

$$v_N = -\langle \Psi_2, \Phi_2 \rangle. \tag{4.20c}$$

Equation (4.20*a*) yields

$$u_N = \ln\langle \Lambda\Psi_2, \Phi_1 \rangle. \tag{4.20d}$$

Similarly, we define

$$v_{N+j} = 2P_{N+j} \qquad j = 1, \ldots, N \tag{4.21a}$$

$$u_{N+j} = \ln \frac{\phi_{1j}}{\psi_{2j}} \qquad j = 1, \ldots, N. \tag{4.21b}$$

Then we have

Proposition 9. *Assume that $\lambda_j, \phi_{ij}, \psi_{ij} \in \mathbb{R}, i = 1, 2, j = 1, \ldots, N$. Introduce the separated variables $u_1, \ldots, u_{2N}$ and $v_1, \ldots, v_{2N}$ by (4.20) and (4.21). If $u_1, \ldots, u_{N-1}$, are single zeros of $\bar{B}(\lambda)$, then $v_1, \ldots, v_{2N}$ and $u_1, \ldots, u_{2N}$ are canonically conjugated, i.e. they satisfy (1.1).*

It follows from (4.9), (4.20*a*), (4.20*d*) and (4.21*b*) that

$$q = -\tfrac{1}{2}\mathrm{e}^{u_N} \tag{4.22a}$$

$$\phi_{1j} = \sqrt{\frac{\mathrm{e}^{u_N+u_{N+j}}R(\lambda_j^2)}{\lambda_j K'(\lambda_j^2)}} \qquad \psi_{2j} = \sqrt{\frac{\mathrm{e}^{u_N-u_{N+j}}R(\lambda_j^2)}{\lambda_j K'(\lambda_j^2)}} \qquad j = 1, \ldots, N. \tag{4.22b}$$

By substituting u_k into (4.13) and using (4.16) and (4.19*c*), one obtains the first $N-1$ separated equations

$$v_k = \bar{A}(u_k) = \frac{2}{u_k}\left[\sqrt{\tilde{P}(u_k)} - \sqrt{P(0)}\right] \qquad k = 1, \ldots, N-1 \tag{4.23a}$$

where $P(0)$ are given by (4.16) and

$$\tilde{P}(\tilde{\lambda}) = 1 + \sum_{j=1}^{N}\left[\frac{P_j}{\tilde{\lambda} - \lambda_j^2} + \frac{\lambda_j^4 P_{N+j}^2}{(\tilde{\lambda} - \lambda_j^2)^2}\right].$$

It follows from (4.15*b*), (4.16) and (4.20*c*) that

$$v_N = 2 - 2\sqrt{P(0)} - 2\sum_{i=1}^{N} P_{N+i}. \tag{4.23b}$$

The separated equations (4.23) and (4.21*a*) may be integrated to give the generating function of the canonical transformation

$$S(u_1, \ldots, u_{2N}) = \sum_{k=1}^{N-1}\left[\int^{u_k} \frac{2}{\tilde{\lambda}}\sqrt{\tilde{P}(\tilde{\lambda})}\,\mathrm{d}\tilde{\lambda} - 2\sqrt{P(0)}\ln|u_k|\right]$$
$$+\left(2 - 2\sqrt{P(0)} - 2\sum_{i=1}^{N} P_{N+i}\right)u_N + 2\sum_{i=1}^{N} P_{N+i}u_{N+i}. \tag{4.24}$$

The Jacobi inversion problem for the FDIHS (4.10) is

$$\sum_{k=1}^{N-1}\left[\int^{u_k}\frac{1}{\tilde\lambda(\tilde\lambda-\lambda_i^2)\sqrt{\tilde P(\tilde\lambda)}}\mathrm{d}\tilde\lambda+\frac{1}{\lambda_i^2\sqrt{P(0)}}\ln|u_k|\right]+\frac{1}{\lambda_i^2\sqrt{P(0)}}u_N=\gamma_i-2x$$

$$\sum_{k=1}^{N-1}\left[\int^{u_k}\frac{\lambda_i^4 P_{N+i}}{\tilde\lambda(\tilde\lambda-\lambda_i^2)^2\sqrt{\tilde P(\tilde\lambda)}}\mathrm{d}\tilde\lambda-\frac{P_{N+i}}{\sqrt{P(0)}}\ln|u_k|\right]$$
$$-\left(\frac{P_{N+i}}{\sqrt{P(0)}}+1\right)u_N+u_{N+i}=\gamma_{N+i}\qquad i=1,\ldots,N. \tag{4.25}$$

The Jacobi inversion problem for the FDIHS (4.11) is

$$\sum_{k=1}^{N-1}\left[\int^{u_k}\frac{1}{\tilde\lambda(\tilde\lambda-\lambda_i^2)\sqrt{\tilde P(\tilde\lambda)}}\mathrm{d}\tilde\lambda+\frac{1}{\lambda_i^2\sqrt{P(0)}}\ln|u_k|\right]+\frac{1}{\lambda_i^2\sqrt{P(0)}}u_N$$
$$=\bar\gamma_i+\left(2\lambda_i^2-\sum_{k=1}^{N}P_k\right)t_2$$

$$\sum_{k=1}^{N-1}\left[\int^{u_k}\frac{\lambda_i^4 P_{N+i}}{\tilde\lambda(\tilde\lambda-\lambda_i^2)^2\sqrt{\tilde P(\tilde\lambda)}}\mathrm{d}\tilde\lambda-\frac{P_{N+i}}{\sqrt{P(0)}}\ln|u_k|\right]$$
$$-\left(\frac{P_{N+i}}{\sqrt{P(0)}}+1\right)u_N+u_{N+i}=\bar\gamma_{N+i}+2\lambda_i^4P_{N+i}t_2\qquad i=1,\ldots,N. \tag{4.26}$$

Finally, since the CDNS equations (4.6) are factorized by the FDIHS (4.10) and (4.11), combining equation (4.25) with (4.26) gives rise to the Jacobi inversion problem for the CDNS equations (4.6),

$$\sum_{k=1}^{N-1}\left[\int^{u_k}\frac{1}{\tilde\lambda(\tilde\lambda-\lambda_i^2)\sqrt{\tilde P(\tilde\lambda)}}\mathrm{d}\tilde\lambda+\frac{1}{\lambda_i^2\sqrt{P(0)}}\ln|u_k|\right]+\frac{1}{\lambda_i^2\sqrt{P(0)}}u_N$$
$$=\gamma_i-2x+\left(2\lambda_i^2-\sum_{k=1}^{N}P_k\right)t_2\qquad i=1,\ldots,N \tag{4.27a}$$

$$\sum_{k=1}^{N-1}\left[\int^{u_k}\frac{\lambda_i^4 P_{N+i}}{\tilde\lambda(\tilde\lambda-\lambda_i^2)^2\sqrt{\tilde P(\tilde\lambda)}}\mathrm{d}\tilde\lambda-\frac{P_{N+i}}{\sqrt{P(0)}}\ln|u_k|\right]$$
$$-\left(\frac{P_{N+i}}{\sqrt{P(0)}}+1\right)u_N+u_{N+i}=\gamma_{N+i}+2\lambda_i^4P_{N+i}t_2\qquad i=1,\ldots,N. \tag{4.27b}$$

If ϕ_{1j},ψ_{2j},q defined by (4.22) can be solved from (4.27) by using the Jacobi inversion technique, then ϕ_{2j},ψ_{1j} can be obtained from the first equation and the last equation in (4.10), respectively. Finally, q and $r=-\langle\Lambda\Psi_1,\Phi_2\rangle$ provide the solution to the CDNS equations (4.6).

In general, the above procedure can be applied to the whole Kaup–Newell hierarchy (4.4). Set

$$A^2(\lambda)+B(\lambda)\,C(\lambda)=\sum_{k=0}^{\infty}\tilde F_k\lambda^{-2k} \tag{4.28a}$$

where $\tilde{F}_k, k = 1, 2, \ldots,$ are also integrals of motion for both the x-FDHSs (4.10) and the t_n-binary constrained flows (2.16). Comparing (4.28a) with (4.13), one obtains

$$\tilde{F}_0 = 1 \qquad \tilde{F}_k = \sum_{j=1}^{N} \left[\lambda_j^{2k-2} P_j + (k-1)\lambda_j^{2k} P_{N+j}^2\right] \qquad k = 1, 2, \ldots. \tag{4.28b}$$

By employing the method in [28, 29], the t_n-FDIHS obtained from the t_n-constrained flow is of the form

$$\Phi_{1,t_n} = \frac{\partial F_n}{\partial \Psi_1} \qquad \Phi_{2,t_n} = \frac{\partial F_n}{\partial \Psi_2} \qquad \Psi_{1,t_n} = -\frac{\partial F_n}{\partial \Phi_1} \qquad \Psi_{2,t_n} = -\frac{\partial F_n}{\partial \Phi_2} \tag{4.29a}$$

with the Hamiltonian

$$F_n = 2\sum_{m=0}^{n-1} \left(-\tfrac{1}{2}\right)^m \frac{\alpha_m}{m+1} \sum_{l_1+\cdots+l_{m+1}=n} \tilde{F}_{l_1} \ldots \tilde{F}_{l_{m+1}} \tag{4.29b}$$

where $l_1 \geqslant 1, \ldots, l_{m+1} \geqslant 1$, and α_m are given by (2.22). Since the nth Kaup–Newell equations (4.4) are factorized by the x-FDIHS (4.10) and the t_n-FDIHS (4.29). We have the following proposition.

Proposition 10. *The Jacobi inversion problem for the nth Kaup–Newell equations (4.4) is given by*

$$\sum_{k=1}^{N-1}\left[\int^{u_k} \frac{1}{\tilde{\lambda}(\tilde{\lambda}-\lambda_i^2)\sqrt{\tilde{P}(\tilde{\lambda})}}\, \mathrm{d}\tilde{\lambda} + \frac{1}{\lambda_i^2\sqrt{P(0)}} \ln|u_k|\right] + \frac{1}{\lambda_i^2\sqrt{P(0)}} u_N$$
$$= \gamma_i - 2x + 2t_n \sum_{m=0}^{n-1} \left(-\tfrac{1}{2}\right)^m \alpha_m \sum_{l_1+\cdots+l_{m+1}=n} \lambda_i^{2l_{m+1}-2} \tilde{F}_{l_1} \ldots \tilde{F}_{l_m} \qquad i = 1, \ldots, N \tag{4.30a}$$

$$\sum_{k=1}^{N-1}\left[\int^{u_k} \frac{\lambda_i^4 P_{N+i}}{\tilde{\lambda}(\tilde{\lambda}-\lambda_i^2)^2\sqrt{\tilde{P}(\tilde{\lambda})}}\, \mathrm{d}\tilde{\lambda} - \frac{P_{N+i}}{\sqrt{P(0)}} \ln|u_k|\right] - \left(\frac{P_{N+i}}{\sqrt{P(0)}} + 1\right) u_N + u_{N+i}$$
$$= \gamma_{N+i} + 2t_n \sum_{m=0}^{n-1} \left(-\tfrac{1}{2}\right)^m \alpha_m \sum_{l_1+\cdots+l_{m+1}=n} (l_{m+1}-1)\lambda_i^{2l_{m+1}} P_{N+i} \tilde{F}_{l_1} \ldots \tilde{F}_{l_m}$$
$$i = 1, \ldots, N \tag{4.30b}$$

where $l_1 \geqslant 1, \ldots, l_{m+1} \geqslant 1$, and $\tilde{F}_{l_1}, \ldots, \tilde{F}_{l_m}$, are given by (4.28b).

For example, the third equations in the Kaup–Newell hierarchy with $n = 3$ are of the form

$$q_{t_3} = -\tfrac{1}{4} q_{xxx} - \tfrac{3}{8}\left(q^3 r^2 + 2qrq_x\right)_x \qquad r_{t_3} = -\tfrac{1}{4} r_{xxx} - \tfrac{3}{8}\left(r^3 q^2 - 2qrr_x\right)_x. \tag{4.31}$$

The Kaup–Newell equations (4.31) can be factorized by the x-FDIHS (4.10) and t_3-FDIHS with the Hamiltonian F_3 defined by

$$F_3 = \sum_{j=1}^{N}\left(2\lambda_j^4 P_j + 4\lambda_j^6 P_{N+j}^2\right) - \left[\sum_{j=1}^{N}\left(\lambda_j^2 P_j + \lambda_j^4 P_{N+j}^2\right)\right] \sum_{j=1}^{N} P_j + \tfrac{1}{4}\left(\sum_{j=1}^{N} P_j\right)^3. \tag{4.32}$$

The Jacobi inversion problem for equations (4.31) is given by

$$\sum_{k=1}^{N-1}\left[\int^{u_k}\frac{1}{\tilde\lambda(\tilde\lambda-\lambda_i^2)\sqrt{\tilde P(\tilde\lambda)}}\,\mathrm{d}\tilde\lambda+\frac{1}{\lambda_i^2\sqrt{P(0)}}\ln|u_k|\right]+\frac{1}{\lambda_i^2\sqrt{P(0)}}u_N$$

$$=\gamma_i-2x+\left[2\lambda_i^4-\sum_{j=1}^{N}\left(\lambda_j^2P_j+\lambda_i^2P_j+\lambda_j^4P_{N+j}^2\right)+\tfrac{3}{4}\left(\sum_{j=1}^{N}P_j\right)^2\right]t_3$$

$$i=1,\ldots,N$$

$$\sum_{k=1}^{N-1}\left[\int^{u_k}\frac{\lambda_i^4P_{N+i}}{\tilde\lambda\left(\tilde\lambda-\lambda_i^2\right)^2\sqrt{\tilde P(\tilde\lambda)}}\,\mathrm{d}\tilde\lambda-\frac{P_{N+i}}{\sqrt{P(0)}}\ln|u_k|\right]-\left(\frac{P_{N+i}}{\sqrt{P(0)}}+1\right)u_N+u_{N+i}$$

$$=\gamma_{N+i}+\left[4\lambda_i^6P_{N+i}-\lambda_i^4P_{N+j}\sum_{j=1}^{N}P_j\right]t_3\qquad i=1,\ldots,N.$$

In general, the method can be applied to all high-order binary constrained flows (4.8) and whole KN hierarchy (4.4) in exactly the same way.

5. Concluding remarks

For high-order binary constrained flows, the method in [1–6] allows us to directly introduce $N+k_0$ pairs of canonical separated variables and $N+k_0$ separated equations via the Lax matrices and the generating function of the integrals of motion. In this paper we propose a new method for determining additional N pairs of canonical separated variables and N additional separated equations for high-order binary constrained flows by directly using N additional integrals of motion. This method is completely different from that proposed in [23, 24] and can be applied to all high-order binary constrained flows and other soliton hierarchies admitting 2×2 Lax pairs.

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